Nonbook problems

1. Verify that for any nonzero constant b, the function $f(x) = \frac{1}{b}\cosh(bx)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - b\sqrt{1 + (\frac{dy}{dx})^2} = 0.$$

(Recall that the function "cosh" is defined by $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

2. Verify that on the interval $-2 < x < 2$, the two continuous functions $y(x) = \sqrt{4-x^2}$ and $y(x) = -\sqrt{4-x^2}$, obtainable from the implicit solution $x^2 + y^2 = 4$ of the differential equation $x + y\ dy/dx = 0$, are (explicit) solutions of this differential equation.

3. Show that if L_1 and L_2 are linear operators, then $L_1 + L_2$ is a linear operator.

4. Show by induction on n that if an operator L is linear, then for all $n \geq 1$, all constants $c_1, c_2, \ldots, c_n$, and all functions $f_1, f_2, \ldots f_n$,

$$L[c_1 f_1 + c_2 f_2 + \ldots + c_n f_n] = c_1 L[f_1] + c_2 L[f_2] + \ldots + c_n L[f_n].$$

5. Show that the following distributive law holds for *linear* operators: if L_1, L_2, R_1, and R_2 are linear operators, then

$$(L_1 + L_2)(R_1 + R_2) = L_1 R_1 + L_1 R_2 + L_2 R_1 + L_2 R_2.$$

Note: Because linear operators don't commute with each other in general (see the next exercise), the formula above is valid *only* with the L's in front of the R's on the right-hand side of the equation.

6. In class it was stated that general linear differential operators L_1, L_2 do not commute with each other: $L_1 L_2 \neq L_2 L_1$. Part (a) of this problem verifies this statement by giving examples of operators that do not commute. It is still true that *some* linear differential operators commute with each other; you will see examples of this in parts (b) and (c).

(a) Let L_1 be the operator "multiplication by p", where p is a *non-constant* function, and let $L_2 = D$ (the first-derivative operator). Show that $L_1 L_2 \neq L_2 L_1$.

(b) Let L_1 and L_2 be as in part (a), but this time assume that p is a *constant* function. Show that in this case $L_1 L_2 = L_2 L_1$.

(c) Let a and b be constants, and let L_1 and L_2 be the linear differential operators $D + a$ and $D + b$, respectively. Using part (b) plus Exercise 5 above, show that $L_1 L_2 = L_2 L_1$.

What you have shown in parts (a), (b) and (c) are special cases of a more general principle: all *constant-coefficient* linear differential operators $a_n D^n + a_{n-1} D^{n-1} + \ldots + a_1 D + a_0$ commute with each other (here the a's are constants), but *non-constant-coefficient* linear differential operators $p_n D^n + p_{n-1} D^{n-1} + \ldots + p_1 D + p_0$ (here the p's are functions at least one of which is non-constant) in general do not commute with constant-coefficient operators or with other non-constant-coefficient operators.

7. The method of problems 4.3/38–39 in the book can be adapted to give solutions to several cases of Cauchy-Euler (pronounced "Co-she Oiler") equations not covered in the book's exercises. In this problem we consider these other cases.

(a) Fix numbers a, b, c (with $a \neq 0$) and consider the second-order homogeneous Cauchy-Euler equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0. \tag{1}$$

If we consider this equation on the interval $\{x < 0\}$, the substitution $x = e^t$ cannot be used (why not?). However, using the Chain Rule and the substitution $x = -u$, show that (1) for $x < 0$ is equivalent to the equation

$$au^2\frac{d^2z}{du^2} + bu\frac{dz}{du} + cz = 0 \tag{2}$$

for $u > 0$, where $z(u) = y(x) = z(-x)$. Except for the names of the variables, equations (1) and (2) are the same. Use this to show that if $\phi(x)$ is a solution of $ax^2y'' + bxy' + cy = 0$ on the interval $\{x > 0\}$, then $\phi(-x)$ is a solution of the same DE on the interval $\{x < 0\}$, and vice-versa. Thus show that if $y_{\text{gen}}(x)$ is the general solution of (1) on the interval $\{x > 0\}$, then $y_{\text{gen}}(|x|)$ is the general solution of (1) on the interval $\{x < 0\}$.

(b) Using the methods of the book's 4.3/38–39, find the general solution $y(x)$ of

$$6x^2y'' + xy' + y = 0 \tag{3}$$

on the interval $\{x > 0\}$. Then, using part (a) above, find the general solution of (3) on the interval $\{x < 0\}$.

(c) Using the methods of the book's 4.3/38–39, find the general solution $y(x)$ of

$$x^2y'' + 5xy' + 4y = 0 \tag{4}$$

on the interval $\{x > 0\}$. (Remember that your answer must be expressed purely in terms of x, not partly in terms of x and partly in terms of t.) Then, using part (a) above, find the general solution of (4) on the interval $\{x < 0\}$.