Elementary Differential Equations—MAP 2302—Fall 2010
Extra Credit Project

Rules. You are not allowed to make this project more difficult for me to grade than it needs to be.

- The work you hand in must be *neat* and *legible.*
- No work in pencil will be accepted. Either write in pen, or type up your work.
- Write (or type) on clean sheets of 8.5" x 11" paper, leaving enough space for me to write comments.
- I will not grade anything that is messy, that has been over-written to correct an error, that even *looks* like it has been erased and then written over, or that has shreds of paper dangling from it (for example, from being ripped out of a spiral-bound notebook).
- *Staple* your sheets together in the upper left-hand corner. **No other method of attachment will be accepted.** I know from experience that if you attach your sheets any other way, they will come apart, making extra work for me. You are all resourceful enough to find a stapler you can use. Don't expect to borrow your friend's stapler in class on the due date. Your friend will be absent, late, or out of staples; even if not, I don't want the first five minutes of class taken up by stapling that should have been done before you entered the classroom.
- Write *sentences*, not just equations. Do not make me guess whether an equation you're writing is something that you're *assuming*, something that's your *goal to show*, or something that you think *follows from what you've previously written.* Observe the way that mathematics is written in your textbook. It's not just "Equation. Equation. Equation." The logical relation of each equation to the discussion, or to previous equations, is made clear. You need to do that too. On a timed exam, I'm willing to make allowances, but in this project you don't have the same sort of time pressure.

If your work is difficult for me to read or understand *for any reason*—messiness, ambiguous or ungrammatical sentences, or anything else—I will not grade past the point at which I first have difficulty.

You are *not* allowed to work with anyone or consult any person other than your professor about this project (except that any discussions you may have had prior to Oct. 31 are okay). You *are* allowed to consult calculus textbooks in case you've forgotten some part of calculus that I refer to. There are no other sources that you should need to use. But if you're curious enough to look up other references on Clairaut equations, *and are able to understand them* (most are written for mathematicians, not students), you will probably learn something, so I don't want to discourage that.

Therefore you're allowed to use other textbooks and online sources, as long as you cite them appropriately. **However, the work that you hand in must be self-contained. Do not just say "X is true" and give a reference to it. If you do what I've just said not to do, I will stop grading at that point. I don't have the time to hunt down what you read, which I would need to do to determine whether you are taking something out of context.** If you want to use a fact that you read somewhere other than a calculus textbook, write the whole argument justifying the fact, not just the conclusion.

Warning: I checked several online sources, and some textbooks, on topics relevant to this project. Some of what's out there, such as the Wikipedia entry on Clairaut equations, has gaps and/or is just plain wrong. I've gone to considerable lengths to lead you to true conclusions reached by valid arguments, stepping you through arguments were missing from the sources that I checked, so I will not accept a short-cut around these arguments with the excuse that you read it somewhere. If you use an argument from some source that's wrong, or that has a gap, you will not get credit for it.

Due date: Monday, November 15. Projects are due at the *start* of class.

This project is a modified version of Group Project E in Chapter 2 of your textbook (Nagle, Saff, and Snider, *Fundamentals of Differential Equations and Boundary Value Problems*, 5th edition). The presentation in the book has some logical gaps that I will step you through.

The project has parts I and II. Part I is worth 17 points (out of 115; approximately 15 percentage points) towards your Midterm 1 score, even if this makes your score exceed 100%. Part II is extra-extra credit. To complete both parts I and II will require more work from you than I felt was reasonable for the original purpose of this project (to allow students who'd made a certain very costly differentiation mistake on Midterm 1 a way to redeem themselves). Part I should not take anyone very long to complete. Part II is more challenging from part (g) on, and I don't expect everyone to have the interest, the time, and skills to complete it.

However, I wanted to include Part II in order to correct some misimpressions that you might get from the book's presentation (particularly from part (b) in the book). Also, when you draw the graphs in the last part of Part I, you will see something that may pique your interest—it may remind you of a type of artwork you may have seen, and you may actually have fun!—and I wanted to give you the opportunity to explore a little further. **Only students who hand in work on all sub-parts of Part I are eligible to hand in anything on Part II.**

Part II, sub-parts (f),(g),(h), and (i) are *together* worth 10 extra-credit *percentage* points towards Midterm 2. (To re-iterate: that's 10 points for all four of these sub-parts *put together*, not 10 points for each sub-part.) **You must hand in all four of these sub-parts for me to grade *any* of them.**

Part II, sub-parts (j) and (k) are *together* worth 5 extra-credit *percentage* points towards Midterm 3. (To re-iterate: that's 5 points for both of these sub-parts *put together*, not 5 points for each sub-part.) **Only students who hand in work on all of the previous sub-parts (a) through (i) are eligible to hand in anything on sub-parts (j) and (k). You must hand in both of the sub-parts (j) and (k) for me to grade *either* of them.**

Part II, sub-part (l) is worth 5 extra-credit *percentage* points towards Midterm 3. **Only students who hand in work on all of the sub-parts (a) through (k) are eligible to hand in anything on sub-part (l).**

You'll notice that I've incorporated a lot of set-up and discussion into the presentation of the project on the following pages. This is just to help you understand what you're doing; it does not mean that *you* have to write anything lengthy. There's actually relatively little writing that you should need to do.

Part I

Set-up and discussion. A *Clairaut equation* is a differential equation of the form

$$y = x\frac{dy}{dx} + f(dy/dx) \tag{1}$$

where f is a one-variable function that is defined on some domain $D \subset \mathbf{R}$ and is continuously differentiable on some open interval $I \subset D$.[1]

(a) **Work for you to do.** Show that for every number $c \in D$, the function ϕ_c defined by $\phi_c(x) = cx + f(c)$ is a solution of (1).

Discussion. Note that the domain of ϕ_c is the whole real line, and that the graph of this solution is the straight line with equation

$$y = cx + f(c). \tag{2}$$

Thus the collection of functions $\{\phi_c \mid c \in D\}$ is a one-parameter family of solutions of (1). We will refer to these as *straight-line solutions* below.

(b) **Set-up and discussion**. Recall from Calculus 2 that parametric equations $x = g(t)$, $y = h(t)$, as t runs over an open interval $\widetilde{I}$, implicitly determine y *uniquely* as a function of x provided that g and h are differentiable on $\widetilde{I}$ and that $g'(t)$ is not zero for any $t \in \widetilde{I}$; moreover, this function $y(x)$ is differentiable. Said another way, under these hypotheses the curve $\mathcal{C}$ traced out by $x = g(t)$, $y = h(t)$, as t runs over $\widetilde{I}$, is the graph of the equation $y = \phi(x)$ for some unique function ϕ—which happens also to be differentiable—defined on the x-interval $\{g(t) \mid t \in \widetilde{I}\}$. We will refer to this latter interval as the *range of* g *on* $\widetilde{I}$. (The Intermediate Value Theorem, about which you may have learned in Calculus 1, can be used to show that the range of g on $\widetilde{I}$ is indeed an interval. The hypothesis that $g'(t) \neq 0$ can be used to show that this interval is open.)

Recall also that, in this case, if we set $y = \phi(x)$ then

$$\frac{dy}{dx} = \frac{h'(t)}{g'(t)} = \frac{dy/dt}{dx/dt} \tag{3}$$

(Here it is understood that $\frac{dy}{dx}$ is evaluated at the point $x = g(t)$, so that we may view $\frac{dy}{dx}$ as a function of t.) Finally, recall that in this situation $\frac{d^2y}{dx^2}$ exists and is given by

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}(t)\right)}{g'(t)} \tag{4}$$

[1]It is not essential that I be open to talk about Clairaut equations. However, if I is not open then more words have to be added, at certain places in the discussion, to deal with endpoints of I and some other intervals. To avoid these technicalities we are making the simplifying assumption here that I is open.

provided that $\frac{d}{dt}\left(\frac{dy}{dx}(t)\right)$ exists.

Work for you to do. Now assume that the function f in (1) is twice differentiable on I and that $f''(t)$ is not zero for any $t \in I$. Consider the curve $\mathcal{C}$ determined by the parametric equations

$$\begin{array}{l} x(t) = -f'(t), \\ y(t) = f(t) - tf'(t), \end{array} \qquad t \in I. \tag{5}$$

Show that the conditions stated above for parametric equations to determine y uniquely as a differentiable function of x—a function that we will call ϕ_{sing}—are met by (5), and that at each point of $\mathcal{C}$, $\frac{dy}{dx} = t$. Then show that ϕ_{sing} is a solution of (1), and that the domain J of ϕ_{sing} is the range of the function $-f'$ on I. Thus ϕ_{sing} is a solution on J of *both* the following differential equations:

$$y = x\frac{dy}{dx} + f\left(\frac{dy}{dx}\right), \tag{6}$$

$$x = -f'\left(\frac{dy}{dx}\right). \tag{7}$$

Set-up and discussion, continued. Because (5) determines $y(x)$ uniquely, ϕ_{sing} is the *only* solution of (6)–(7) on J with $y'(x) \in I$ for all $x \in J$. Even more strongly, if $J_1 \subset J$ is any subinterval, ϕ_{sing} is the only solution of (6)–(7) on J_1 with the property that $y'(x) \in I$ for all $x \in J_1$.

The function ϕ_{sing} is called a *singular solution* of (1), and may be called "*the* singular solution on J with slopes in I". We may call ϕ_{sing} simply "*the* singular solution" (without mentioning I and J) when the following uniqueness-guaranteeing condition is met:

> **Condition U**. The set $\{t_0 \mid f''(t_0) \text{ exists and is not } 0\}$ is an open interval, and we take I in (5) to be this interval.

In the term "singular solution", the word "singular" is used to indicate that there is something very unusual about this solution. You will see shortly just what's so unusual.

(c) **Work for you to do.** Show that for every point (x_0, y_0) on the graph $\mathcal{C}$ of ϕ_{sing}, the line tangent to $\mathcal{C}$ at (x_0, y_0) is the graph of the straight-line solution ϕ_c (as defined in part (a)) with $c = \phi'_{\text{sing}}(x_0)$.

Information. Because of this property—being tangent at each point to one member of a given one-parameter family of curves—the curve $\mathcal{C}$ is called the *envelope* of the graphs of the straight-line solutions.

(d) **Work for you to do.** Using part (c), show that for every point (x_0, y_0) on the graph $\mathcal{C}$ of ϕ_{sing}, the initial-value problem

$$y = x\frac{dy}{dx} + f(dy/dx), \quad y(x_0) = y_0 \tag{8}$$

has at least two solutions. Why does this not contradict the Fundamental Existence and Uniqueness Theorem (Theorem 1 on p. 12 of your textbook)?

Discussion, continued. In an old homework exercise you saw that there are differential equations for which there is *some* initial-condition point (x_0, y_0) for which the corresponding initial-value problem has more than one solution. However, in all examples of this behavior that you have seen in this class, there was only one "bad" point (x_0, y_0). More generally, there may be more than one such point, but typically such points are isolated from each other. What is so unusual about the solution ϕ_{sing} is that uniqueness-of-solution-to-IVP is violated at *every* point of its graph. This is the reason we use the term "singular solution".

(e) **Work for you to do.** For the Clairaut equation

$$y = x\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 \tag{9}$$

(this is the equation on p. 88 of your textbook, part (d)), do the following:

1. Show that Condition U is met, and use the methods of parts (a) and (b) to find the family of straight-line solutions and the singular solution.

2. For the singular solution, eliminate the parameter t by using the top line of (5) to solve for t in terms of x, and re-express the singular solution *explicitly* in the form $y = \phi_{\text{sing}}(x)$ (i.e. find a formula for $\phi_{\text{sing}}(x)$).

3. *Carefully* graph at least 10 of the straight-line solutions with positive slope and at least 10 with negative slope, as well as the singular solution, using a single set of xy axes (i.e. produce one picture with all 21 [or more] graphs, so that you can see visually how the graphs relate to each other). Either use graph paper and a ruler to draw the straight-line graphs, or, if you have enough facility with a computer, you may have a computer generate the graphs. Your graph of the singular solution doesn't have to be as accurate as the straight-line graphs, since you can't just draw it with a ruler. Your picture should be consistent with what was stated in part (c) about the relation between the singular solution and the straight-line solutions.

Part II

(f) **The sliding ladder.** A ladder of length L is positioned in the first quadrant of the xy plane with one end on the positive x-axis and the other end on the y-axis. (We are viewing the ladder edge-on, so that it looks like a line segment.) Obviously there are infinitely many such positions for the ladder, of varying steepness; if we constrain the top of the ladder to move in a track on the y-axis, grease the x-axis, and start the ladder at some initial position, it will slide down through the various less-steep positions until it settles on the ground.

Work for you to do.

1. Show that if the acute angle between the ladder and the x-axis is θ, then the ladder lies along the first-quadrant portion of the graph of

$$y = -(\tan\theta)x + L\sin\theta. \tag{10}$$

2. Letting $c = -\tan\theta$ (where $0 < \theta < \frac{\pi}{2}$, so that $-\infty < c < 0$), show that (10) can be rewritten as

$$y = cx - L\frac{c}{\sqrt{1+c^2}}\ . \tag{11}$$

   (Although $c = 0$ gives a perfectly good position of the ladder, we exclude it from consideration above for reasons mentioned earlier in a footnote.)

3. Comparing (11) with (2), we see that the positions of the ladder are the (first-quadrant portions of the) straight-line solutions of the Clairaut equation (1) with

$$f(t) = -L\frac{t}{\sqrt{1+t^2}}, \quad -\infty < t < 0. \tag{12}$$

   Check (by explicit calculation) that $f''(t)$ exists and is nowhere zero on $I = (-\infty, 0)$, and show that the range of $-f'$ on I is $J = (0, L)$. Then find, in parametric form, the singular solution of this Clairaut equation on $(0, L)$ with slopes in $(-\infty, 0)$.

4. Eliminate the parameter t and re-express this singular solution *explicitly* in the form $y = \phi_{\mathrm{sing}}(x)$. Show by direct calculation that this explicit function ϕ_{sing} is a solution of (1) with f as in (12). What is the domain of ϕ_{sing}?

5. Taking $L = 10$, *carefully* graph the first-quadrant portions of at least 15 of the straight-line solutions, along with the singular solution, using a single set of xy axes. As in part (e)(3) of this project, either use graph paper and a ruler to draw the straight-line graphs, or have a computer generate the graphs. Your graph of the singular solution doesn't have to be as accurate as the straight-line graphs. Your picture should be consistent with what was stated in part (c).

(g) **Work for you to do.** For the Clairaut equation in part (f), find another singular solution on $(0, L)$ but with slopes in $(0, \infty)$. Graph it and the corresponding straight-line solutions, analogously to what you did in part (f)(5) above.

Further set-up and discussion. In parts (a) and (b) of this problem you showed that equation (1) has both the one-parameter family of straight-line solutions, and the singular solution ϕ_{sing} on J with slopes in I (where J is the range of $-f'$ on I). The next question is: are there any other solutions? This is what the remainder of this problem addresses. (Note: our f now is not the specific function defined in (12); it is a general function that is twice differentiable on I, with $f''(t)$ nowhere zero on I.)

(h) Let t_0 be any value in I, let $c = f(t_0)$, and let $x_0 = -f'(t_0)$. For $x \in J$ define

$$\begin{aligned} \phi_+(x) &= \begin{cases} \phi_c(x) & \text{if } x \le x_0, \\ \phi_{\text{sing}}(x) & \text{if } x > x_0, \end{cases} \\ \phi_-(x) &= \begin{cases} \phi_{\text{sing}}(x) & \text{if } x \le x_0, \\ \phi_c(x) & \text{if } x > x_0. \end{cases} \end{aligned}$$

Work for you to do. Show that ϕ_+ and ϕ_- are differentiable on J and that they satisfy (1).

Discussion. You've now shown that (1) admits solutions that are neither straight-line solutions nor the singular solution, but "hybrids" whose graphs consist of a piece of a straight-line solution joined smoothly to a piece of the graph of the singular solution. (These extra solutions are not mentioned in your textbook.)

(i) **Work for you to do.** The "hybrid" solutions in part (h) had one "break-point" x_0 in the domain (where "break-point" just means that we used one formula to the left of x_0 and another formula to the right; the graph was still smooth at x_0). Can you find a hybrid solution with two break-points? More than two?

(j) **Work for you to do.** Using (4), show that $\phi''_{\text{sing}}(x)$ exists for all $x \in J$. Show, however, that for the hybrid solution ϕ_+ in part (h), ϕ''_+ has a "jump discontinuity" at x_0: the one-sided limits $\lim_{x \to x_0^-} \phi''_+(x)$ and $\lim_{x \to x_0^+} \phi''_+(x)$ both exist, but are not equal. (The same is true of ϕ_-, but since the argument is virtually identical to the one for ϕ_+, I don't want you to hand in the argument for ϕ_-.)

(k) **Discussion and set-up.** Using the Mean Value Theorem from Calculus 1, it can be shown that if the derivative of a function has a jump-discontinuity at a point x_0, then the derivative of the function does not exist at x_0. Therefore $\phi''_+(x_0)$ does not exist (a fact that can be shown in other ways as well).

Parts (h) and (j) therefore motivate the following question: are there any *twice-differentiable* solutions of (1) on J, with slopes in I, other than the straight-line solutions and the singular solution? For technical reasons this question is tricky to address. However, if we modify the question slightly, we get one that we can answer completely: are there any twice-*continuously*-differentiable solutions of (1) on J, with slopes in I, other than the straight-line solutions and the singular solution? ("Twice continuously differentiable on J" means that the second derivative exists and is continuous at every point of J. You showed above that ϕ_{sing} is twice continuously differentiable on J. Every straight-line solution has second derivative identically zero, hence is twice continuously differentiable everywhere. Note that even were $\phi''_{+}(x_0)$ to exist, the non-equality of the one-sided limits of ϕ'' at x_0 would imply that ϕ''_{+} is not continuous at x_0.)

To answer this question, proceed through the steps below.

1. **Work for you to do.** Show that every twice-differentiable solution of (1) on J satisfies

$$[x + f'(dy/dx)]\,\frac{d^2y}{dx^2} \equiv 0 \quad \text{on } J. \tag{13}$$

(The hypothesis of twice-differentiability is omitted from your textbook's presentation of this project. Part (h) above shows that the hypothesis matters.)

2. **Work for you to do.** Show that if $y = \phi(x)$ is a twice-differentiable solution of (1) on J that satisfies $\frac{d^2y}{dx^2} = 0$, then ϕ is one of the straight-line solutions ϕ_c defined in part (a).

3. **Set-up and discussion.** If a solution y of (1) is twice *continuously* differentiable, then not only do we know that (13) holds; we know that the factors on the left-hand side, $[x + f'(dy/dx)]$ and $\frac{d^2y}{dx^2}$, are continuous functions of x.

   Can we conclude from this that one of the factors must be identically zero? Not so fast! It is *not* true that if the product of two continuous functions on an interval is identically zero, then one of the functions must be identically zero. For example, the functions q, r defined on $(-\infty, \infty)$ by

$$\begin{aligned} q(x) &= \begin{cases} x^2 & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases} \\ r(x) &= \begin{cases} 0 & \text{if } x \ge 0, \\ x^2 & \text{if } x < 0. \end{cases} \end{aligned}$$

   are continuous—in fact, differentiable—and satisfy $q(x)r(x) = 0$ for all $x \in \mathbf{R}$ even though neither factor is identically zero. (Note that without the continuity assumption on the factors, there would be even simpler counterexamples, such as $\tilde{q}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x \le 0, \end{cases}$ $\tilde{r}(x) = \begin{cases} 0 & \text{if } x > 0, \\ 1 & \text{if } x \le 0. \end{cases}$) If we replace x^2 by x^{100} in the definitions of $q(x)$ and $r(x)$, we get functions that are 99-times differentiable and whose product is identically zero, even though neither function is identically zero. We can even find a pair of *infinitely* differentiable functions with this property (the most common example is given by replacing x^2 above by e^{-1/x^2}, but it takes a little work to show that the resulting functions q and r are infinitely differentiable). So we cannot conclude from (13) *alone* that if y is a twice-continuously-differentiable solution of (1) on J then either $x + f'(dy/dx) \equiv 0$ on J or that $\frac{d^2y}{dx^2} \equiv 0$ on J.

However, we can argue as follows. Suppose that $y = \phi(x)$ is a twice-continuously-differentiable solution of (1) on J with slopes in I. We saw in step 2 above that if this solution satisfies $\frac{d^2y}{dx^2} = 0$—i.e. if $\phi'' \equiv 0$ on J—then ϕ is one of the straight-line solutions ϕ_c defined in part (a). So, to see what *other* solutions there could be, let's assume that $\frac{d^2y}{dx^2}$ is *not* identically 0 on J. Then there is some point $x_0 \in J$ at which $\frac{d^2y}{dx^2} \neq 0$; i.e. $\phi''(x_0) \neq 0$. Because we have assumed that ϕ'' is continuous, this implies there is some open interval $J_1 = (x_0 - \delta, x_0 + \delta) = (a_1, b_1) \subset J$ for which $\phi''(x) \neq 0$ for every $x \in J_1$.

From (13), it then follows that $x + f'(dy/dx) = 0$ identically on J_1. As we observed in part (b), ϕ_{sing} is the only function on J_1 with slopes in I that satisfies both of the equations (6)–(7). Therefore ϕ is identically equal to ϕ_{sing} *on the subinterval* J_1. The question is: are ϕ and ϕ_{sing} identically equal on the whole interval J?[2]

To answer this, suppose the right-endpoint b_1 of J_1 lies in J. (The only way this can fail to happen is if J does not extend to $+\infty$, and b_1 is right-endpoint of J.)

Work for you to do. In part (j), you (should have) produced a formula for ϕ''_{sing} in terms of t. Use this formula to show that $\phi''_{\text{sing}}(b_1) \neq 0$. (Note: b_1 is an x-value, not a t-value.) Use this to show that $\phi''(b_1) \neq 0$ as well.

4. **Discussion.** Continuing from above, since $\phi''(b_1) \neq 0$ and ϕ'' is continuous, and b_1 lies in the open interval J, ϕ'' must remain nonzero slightly to the right of b_1 within J. Therefore we can enlarge J_1 by sliding its right-endpoint slightly to the right, and obtain a new, larger open interval J_2, with right endpoint $b_2 > b_1$, such that ϕ'' is nowhere 0 on J_2. Then, by the same argument as in the "Set-up and discussion" part of step 3, $\phi \equiv \phi_{\text{sing}}$ on J_1.

[2] One of the possibilities we are trying to rule out is the possibility that ϕ may link up with a straight-line solution to produce a "hybrid" solution of the type discussed in parts (h) and (i) of this problem. But just as importantly—if not more importantly—we are trying to rule out solutions *that we may not even have thought of.*

We can continue in this fashion, getting ever-larger intervals of the form (a_1, b) on which $\phi \equiv \phi_{\text{sing}}$. For the set of b's with $b > a_1$ and with the property that $\phi \equiv \phi_{\text{sing}}$ on (a_1, b), there are only two possibilities: either every $b \in J$ with $b > a_1$ has this property—in which case $\phi(x) = \phi_{\text{sing}}(x)$ for every $x \in J$ with $x > a_1$—or there is some largest number $b_{\text{final}} \in J$ for which $\phi \equiv \phi_{\text{sing}}$ on (a_1, b_{final}). In the latter case, the same argument as in the previous paragraph, with b_1 replaced by b_{final}, shows that b_{final} can be moved further to the right, a contradiction; hence this latter case is actually not possible.

Therefore $\phi(x) = \phi_{\text{sing}}(x)$ for every $x \in J$ with $x > a_1$.

Work for you to do. Show how we can now play a similar game, moving a_1 to the left, and reach the conclusion that $\phi \equiv \phi_{\text{sing}}$ on the whole interval J.

Discussion. What you have now shown is that if ϕ is a twice-continuously-differentiable solution of (1) on J, then either ϕ is one of the straight-line solutions ϕ_c, or ϕ is the singular solution ϕ_{sing}.

(l) **Set-up and discussion.** For the Clairaut equation you were working with in part (g), you saw that there were two singular solutions, one with slopes in $(-\infty, 0)$ and one with slopes in $(0, \infty)$. The graphs of both solutions have an "open endpoint" at $(L, 0)$. If you graph the two singular solutions on the same set of axes, and add the endpoint $(L, 0)$ to these graphs, then the graphs of join together to form a single piecewise-smooth curve with a cusp at $(L, 0)$.

The Clairaut equation

$$y = x\frac{dy}{dx} + \cos\left(\frac{dy}{dx}\right) \tag{14}$$

exhibits this sort of exotic behavior gone wild.

Work for you to do:

1. By explicitly finding all the singular solutions, show that (14) has *infinitely many* singular solutions.

2. One (and only one) of these singular solutions has a graph that is symmetric about the y-axis. Graph this singular solution, along with enough of the associated straight-line solutions that the "envelope property" is visible. (The same general instructions as for the other graphs in this project apply here.)

3. Show that if you include the endpoints of the graphs of the all the singular solutions, then the graphs join together to form a piecewise smooth curve with a cusp at each endpoint of each of the singular-solution graphs. Graph several of the singular solutions joined together this way. You are allowed to have a computer generate the graphs.