CHAPTER

1 Introduction

1.1 Background

In the sciences and engineering, mathematical models are developed to aid in the understanding of physical phenomena. These models often yield an equation that contains some derivatives of an unknown function. Such an equation is called a **differential equation.** Two examples of models developed in calculus are the free fall of a body and the decay of a radioactive substance.

In the case of free fall, an object is released from a certain height above the ground and falls under the force of gravity.† Newton's second law, which states that an object's mass times its acceleration equals the total force acting on it, can be applied to the falling object. This leads to the equation (see Figure 1.1)

$$m\frac{d^2h}{dt^2} = -mg\,,$$

where m is the mass of the object, h is the height above the ground, d^2h/dt^2 is its acceleration, g is the (constant) gravitational acceleration, and $-mg$ is the force due to gravity. This is a differential equation containing the second derivative of the unknown height h as a function of time.

Fortunately, the above equation is easy to solve for h. All we have to do is divide by m and integrate twice with respect to t. That is,

$$\frac{d^2h}{dt^2} = -g\,,$$

so

$$\frac{dh}{dt} = -gt + c_1$$

and

$$h = h(t) = \frac{-gt^2}{2} + c_1t + c_2\,.$$

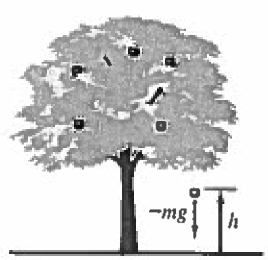

Figure 1.1 Apple in free fall

†We are assuming here that gravity is the *only* force acting on the object and that this force is constant. More general models would take into account other forces, such as air resistance.

We will see that the constants of integration, c_1 and c_2, are determined if we know the *initial* height and the *initial* velocity of the object. We then have a formula for the height of the object at time t.

In the case of radioactive decay (Figure 1.2), we begin from the premise that the rate of decay is proportional to the amount of radioactive substance present. This leads to the equation

$$\frac{dA}{dt} = -kA\,, \qquad k > 0\,,$$

where $A(>0)$ is the unknown amount of radioactive substance present at time t and k is the proportionality constant. To solve this differential equation, we rewrite it in the form

$$\frac{1}{A}dA = -k\,dt$$

and integrate to obtain

$$\int \frac{1}{A}dA = \int -k\,dt$$

$$\ln A + C_1 = -kt + C_2\,.$$

Solving for A yields

$$A = A(t) = e^{\ln A} = e^{-kt}e^{C_2 - C_1} = Ce^{-kt},$$

where C is the combination of integration constants $e^{C_2 - C_1}$. The value of C, as we will see later, is determined if the *initial* amount of radioactive substance is given. We then have a formula for the amount of radioactive substance at any future time t.

Even though the above examples were easily solved by methods learned in calculus, they do give us some insight into the study of differential equations in general. First, notice that the solution of a differential equation is a *function*, like $h(t)$ or $A(t)$, not merely a number. Second, integration[†] is an important tool in solving differential equations (not surprisingly!). Third, we cannot expect to get a unique solution to a differential equation, since there will be arbitrary "constants of integration." The second derivative d^2h/dt^2 in the free-fall equation gave rise to two constants, c_1 and c_2, and the first derivative in the decay equation gave rise, ultimately, to one constant, C.

Whenever a mathematical model involves the **rate of change** of one variable with respect to another, a differential equation is apt to appear. Unfortunately, in contrast to the examples for free fall and radioactive decay, the differential equation may be very complicated and difficult to analyze.

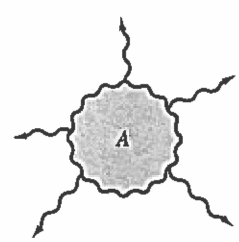

Figure 1.2 Radioactive decay

[†]For a review of integration techniques, see Appendix A.

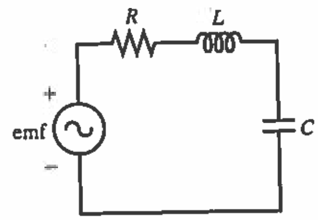

Figure 1.3 Schematic for a series RLC circuit

Differential equations arise in a variety of subject areas, including not only the physical sciences but also such diverse fields as economics, medicine, psychology, and operations research. We now list a few specific examples.

1. In banking practice, if $P(t)$ is the number of dollars in a savings bank account that pays a yearly interest rate of $r\%$ *compounded continuously*, then P satisfies the differential equation

$$\frac{dP}{dt} = \frac{r}{100}P\,, \quad t \text{ in years.} \tag{1}$$

2. A classic application of differential equations is found in the study of an electric circuit consisting of a resistor, an inductor, and a capacitor driven by an electromotive force (see Figure 1.3). Here an application of Kirchhoff's laws† leads to the equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)\,, \tag{2}$$

where L is the inductance, R is the resistance, C is the capacitance, $E(t)$ is the electromotive force, $q(t)$ is the charge on the capacitor, and t is the time.

3. In psychology, one model of the learning of a task involves the equation

$$\frac{dy/dt}{y^{3/2}(1-y)^{3/2}} = \frac{2p}{\sqrt{n}}\,. \tag{3}$$

Here the variable y represents the learner's skill level as a function of time t. The constants p and n depend on the individual learner and the nature of the task.

4. In the study of vibrating strings and the propagation of waves, we find the *partial* differential equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0\,,^{\ddagger} \tag{4}$$

where t represents time, x the location along the string, c the wave speed, and u the displacement of the string, *which is a function of time and location.*

---

†We will discuss Kirchhoff's laws in Section 3.5.

‡*Historical Footnote:* This partial differential equation was first discovered by Jean le Rond d'Alembert (1717–1783) in 1747.

To begin our study of differential equations, we need some common terminology. If an equation involves the derivative of one variable with respect to another, then the former is called a **dependent variable** and the latter an **independent variable.** Thus, in the equation

$$\frac{d^2x}{dt^2} + a\frac{dx}{dt} + kx = 0 \,, \tag{5}$$

t is the independent variable and x is the dependent variable. We refer to a and k as **coefficients** in equation (5). In the equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = x - 2y \,, \tag{6}$$

x and y are independent variables and u is the dependent variable.

A differential equation involving only ordinary derivatives with respect to a single independent variable is called an **ordinary differential equation.** A differential equation involving partial derivatives with respect to more than one independent variable is a **partial differential equation.** Equation (5) is an ordinary differential equation, and equation (6) is a partial differential equation.

The **order** of a differential equation is the order of the highest-order derivatives present in the equation. Equation (5) is a second-order equation because d^2x/dt^2 is the highest-order derivative present. Equation (6) is a first-order equation because only first-order partial derivatives occur.

It will be useful to classify ordinary differential equations as being either linear or nonlinear. Remember that lines (in two dimensions) and planes (in three dimensions) are especially easy to visualize, when compared to nonlinear objects such as cubic curves or quadric surfaces. For example, all the points on a line can be found if we know just two of them. Correspondingly, *linear* differential equations are more amenable to solution than nonlinear ones. Observe that the equations for lines $ax + by = c$ and planes $ax + by + cz = d$ have the feature that the variables appear in *additive combinations of their first powers only*. By analogy a **linear differential equation** is one in which the dependent variable y and its derivatives appear in additive combinations of their first powers.

More precisely, a differential equation is **linear** if it has the format

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = F(x) \,, \tag{7}$$

where $a_n(x), a_{n-1}(x), \ldots, a_0(x)$ and $F(x)$ depend only on the independent variable x. The additive combinations are permitted to have multipliers (coefficients) that depend on x; no restrictions are made on the nature of this x-dependence. If an ordinary differential equation is not linear, then we call it **nonlinear.** For example,

$$\frac{d^2y}{dx^2} + y^3 = 0$$

is a nonlinear second-order ordinary differential equation because of the y^3 term, whereas

$$t^3\frac{dx}{dt} = t^3 + x$$

is linear (despite the t^3 terms). The equation

$$\frac{d^2y}{dx^2} - y\frac{dy}{dx} = \cos x$$

is nonlinear because of the $y\,dy/dx$ term.

Although the majority of equations one is likely to encounter in practice fall into the *nonlinear* category, knowing how to deal with the simpler linear equations is an important first step (just as tangent lines help our understanding of complicated curves by providing local approximations).

1.1 EXERCISES

In Problems 1–12, a differential equation is given along with the field or problem area in which it arises. Classify each as an ordinary differential equation (ODE) or a partial differential equation (PDE), give the order, and indicate the independent and dependent variables. If the equation is an ordinary differential equation, indicate whether the equation is linear or nonlinear.

1. $5\dfrac{d^2x}{dt^2}+4\dfrac{dx}{dt}+9x=2\cos 3t$

   (mechanical vibrations, electrical circuits, seismology)

2. $\dfrac{d^2y}{dx^2}-2x\dfrac{dy}{dx}+2y=0$

   (Hermite's equation, quantum-mechanical harmonic oscillator)

3. $\dfrac{dy}{dx}=\dfrac{y(2-3x)}{x(1-3y)}$

   (competition between two species, ecology)

4. $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

   (Laplace's equation, potential theory, electricity, heat, aerodynamics)

5. $y\left[1+\left(\dfrac{dy}{dx}\right)^2\right]=C$, where C is a constant

   (brachistochrone problem,† calculus of variations)

6. $\dfrac{dx}{dt}=k(4-x)(1-x)$, where k is a constant

   (chemical reaction rates)

7. $\dfrac{dp}{dt}=kp(P-p)$, where k and P are constants

   (logistic curve, epidemiology, economics)

8. $\sqrt{1-y}\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}=0$

   (Kidder's equation, flow of gases through a porous medium)

9. $x\dfrac{d^2y}{dx^2}+\dfrac{dy}{dx}+xy=0$

   (aerodynamics, stress analysis)

10. $8\dfrac{d^4y}{dx^4}=x(1-x)$

    (deflection of beams)

11. $\dfrac{\partial N}{\partial t}=\dfrac{\partial^2 N}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial N}{\partial r}+kN$, where k is a constant

    (nuclear fission)

12. $\dfrac{d^2y}{dx^2}-0.1(1-y^2)\dfrac{dy}{dx}+9y=0$

    (van der Pol's equation, triode vacuum tube)

In Problems 13–16, write a differential equation that fits the physical description.

13. The rate of change of the population p of bacteria at time t is proportional to the population at time t.
14. The velocity at time t of a particle moving along a straight line is proportional to the fourth power of its position x.
15. The rate of change in the temperature T of coffee at time t is proportional to the difference between the temperature M of the air at time t and the temperature of the coffee at time t.
16. The rate of change of the mass A of salt at time t is proportional to the square of the mass of salt present at time t.

17. **Drag Race.** Two drivers, Alison and Kevin, are participating in a drag race. Beginning from a standing start, they each proceed with a constant acceleration. Alison covers the last 1/4 of the distance in 3 seconds, whereas Kevin covers the last 1/3 of the distance in 4 seconds. Who wins and by how much time?

†*Historical Footnote:* In 1630 Galileo formulated the brachistochrone problem ($\beta\rho\acute{\alpha}\chi\iota\sigma\tau o\varsigma$ = shortest, $\chi\rho\acute{o}\nu o\varsigma$ = time), that is, to determine a path down which a particle will fall from one given point to another in the shortest time. It was reproposed by John Bernoulli in 1696 and solved by him the following year.

1.2 Solutions and Initial Value Problems

An nth-order ordinary differential equation is an equality relating the independent variable to the nth derivative (and usually lower-order derivatives as well) of the dependent variable. Examples are

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=x^3 \text{ (second-order, } x \text{ independent, } y \text{ dependent)}$$

$$\sqrt{1-\left(\frac{d^2y}{dt^2}\right)}-y=0 \text{ (second-order, } t \text{ independent, } y \text{ dependent)}$$

$$\frac{d^4x}{dt^4}=xt \text{ (fourth-order, } t \text{ independent, } x \text{ dependent).}$$

Thus, a general form for an nth-order equation with x independent, y dependent, can be expressed as

$$F\left(x,y,\frac{dy}{dx},\ldots,\frac{d^ny}{dx^n}\right)=0, \tag{1}$$

where F is a function that depends on x, y, and the derivatives of y up to order n; that is, on x, $y, \ldots, d^ny/dx^n$. We assume that the equation holds for all x in an open interval I ($a<x<b$, where a or b could be infinite). In many cases we can isolate the highest-order term d^ny/dx^n and write equation (1) as

$$\frac{d^ny}{dx^n}=f\left(x,y,\frac{dy}{dx},\ldots,\frac{d^{n-1}y}{dx^{n-1}}\right), \tag{2}$$

which is often preferable to (1) for theoretical and computational purposes.

Explicit Solution

Definition 1. A function $\phi(x)$ that when substituted for y in equation (1) [or (2)] satisfies the equation for all x in the interval I is called an **explicit solution** to the equation on I.

Example 1 Show that $\phi(x)=x^2-x^{-1}$ is an explicit solution to the linear equation

$$\frac{d^2y}{dx^2}-\frac{2}{x^2}y=0, \tag{3}$$

but $\psi(x)=x^3$ is not.

Solution The functions $\phi(x)=x^2-x^{-1}$, $\phi'(x)=2x+x^{-2}$, and $\phi''(x)=2-2x^{-3}$ are defined for all $x\neq 0$. Substitution of $\phi(x)$ for y in equation (3) gives

$$(2-2x^{-3})-\frac{2}{x^2}(x^2-x^{-1})=(2-2x^{-3})-(2-2x^{-3})=0.$$

Since this is valid for any $x \neq 0$, the function $\phi(x) = x^2 - x^{-1}$ is an explicit solution to (3) on $(-\infty, 0)$ and also on $(0, \infty)$.

For $\psi(x) = x^3$ we have $\psi'(x) = 3x^2$, $\psi''(x) = 6x$, and substitution into (3) gives

$$6x - \frac{2}{x^2}x^3 = 4x = 0\,,$$

which is valid only at the point $x = 0$ and not on an interval. Hence $\psi(x)$ is not a solution. ◆

Example 2 Show that for *any* choice of the constants c_1 and c_2, the function

$$\phi(x) = c_1e^{-x} + c_2e^{2x}$$

is an explicit solution to the linear equation

$$\text{(4)} \qquad y'' - y' - 2y = 0\,.$$

Solution We compute $\phi'(x) = -c_1e^{-x} + 2c_2e^{2x}$ and $\phi''(x) = c_1e^{-x} + 4c_2e^{2x}$. Substitution of ϕ, ϕ', and ϕ'' for y, y', and y'' in equation (4) yields

$$\begin{aligned}&(c_1e^{-x} + 4c_2e^{2x}) - (-c_1e^{-x} + 2c_2e^{2x}) - 2(c_1e^{-x} + c_2e^{2x})\\ &\quad = (c_1 + c_1 - 2c_1)e^{-x} + (4c_2 - 2c_2 - 2c_2)e^{2x} = 0\,.\end{aligned}$$

Since equality holds for all x in $(-\infty, \infty)$, then $\phi(x) = c_1e^{-x} + c_2e^{2x}$ is an explicit solution to (4) on the interval $(-\infty, \infty)$ for any choice of the constants c_1 and c_2. ◆

As we will see in Chapter 2, the methods for solving differential equations do not always yield an explicit solution for the equation. We may have to settle for a solution that is defined implicitly. Consider the following example.

Example 3 Show that the relation

$$\text{(5)} \qquad y^2 - x^3 + 8 = 0$$

implicitly defines a solution to the nonlinear equation

$$\text{(6)} \qquad \frac{dy}{dx} = \frac{3x^2}{2y}$$

on the interval $(2, \infty)$.

Solution When we solve (5) for y, we obtain $y = \pm\sqrt{x^3 - 8}$. Let's try $\phi(x) = \sqrt{x^3 - 8}$ to see if it is an explicit solution. Since $d\phi/dx = 3x^2/\left(2\sqrt{x^3 - 8}\right)$, both ϕ and $d\phi/dx$ are defined on $(2, \infty)$. Substituting them into (6) yields

$$\frac{3x^2}{2\sqrt{x^3 - 8}} = \frac{3x^2}{2\left(\sqrt{x^3 - 8}\right)}\,,$$

which is indeed valid for all x in $(2, \infty)$. [You can check that $\psi(x) = -\sqrt{x^3 - 8}$ is also an explicit solution to (6).] ◆

Implicit Solution

Definition 2. A relation $G(x, y) = 0$ is said to be an **implicit solution** to equation (1) on the interval I if it defines one or more explicit solutions on I.

Example 4 Show that

(7) $$x + y + e^{xy} = 0$$

is an implicit solution to the nonlinear equation

(8) $$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0 .$$

Solution First, we observe that we are unable to solve (7) directly for y in terms of x alone. However, for (7) to hold, we realize that any change in x requires a change in y, so we expect the relation (7) to define implicitly at least one function $y(x)$. This is difficult to show directly but can be rigorously verified using the **implicit function theorem**† of advanced calculus, which guarantees that such a function $y(x)$ exists that is also differentiable (see Problem 30).

Once we know that y is a differentiable function of x, we can use the technique of implicit differentiation. Indeed, from (7) we obtain on differentiating with respect to x and applying the product and chain rules,

$$\frac{d}{dx}(x + y + e^{xy}) = 1 + \frac{dy}{dx} + e^{xy}\left(y + x\frac{dy}{dx}\right) = 0$$

or

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0 ,$$

which is identical to the differential equation (8). Thus, relation (7) is an implicit solution on some interval guaranteed by the implicit function theorem. ◆

Example 5 Verify that for every constant C the relation $4x^2 - y^2 = C$ is an implicit solution to

(9) $$y\frac{dy}{dx} - 4x = 0 .$$

Graph the solution curves for $C = 0, \pm 1, \pm 4$. (We call the collection of all such solutions a *one-parameter family of solutions.*)

Solution When we implicitly differentiate the equation $4x^2 - y^2 = C$ with respect to x, we find

$$8x - 2y\frac{dy}{dx} = 0 ,$$

†See *Vector Calculus*, 6th ed, by J. E. Marsden and A. J. Tromba (Freeman, San Francisco, 2013).

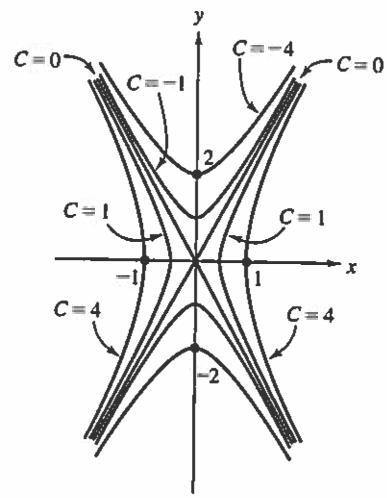

Figure 1.4 Implicit solutions $4x^2 - y^2 = C$

which is equivalent to (9). In Figure 1.4 we have sketched the implicit solutions for $C = 0, \pm 1, \pm 4$. The curves are hyperbolas with common asymptotes $y = \pm 2x$. Notice that the implicit solution curves (with C arbitrary) fill the entire plane and are nonintersecting for $C \neq 0$. For $C = 0$, the implicit solution gives rise to the two explicit solutions $y = 2x$ and $y = -2x$, both of which pass through the origin. ◆

For brevity we hereafter use the term *solution* to mean either an explicit or an implicit solution.

In the beginning of Section 1.1, we saw that the solution of the *second*-order free-fall equation invoked two arbitrary constants of integration c_1, c_2:

$$h(t) = \frac{-gt^2}{2} + c_1 t + c_2 ,$$

whereas the solution of the *first*-order radioactive decay equation contained a single constant C:

$$A(t) = Ce^{-kt} .$$

It is clear that integration of the simple *fourth*-order equation

$$\frac{d^4y}{dx^4} = 0$$

brings in four undetermined constants:

$$y(x) = c_1x^3 + c_2x^2 + c_3x + c_4 .$$

It will be shown later in the text that in general the methods for solving nth-order differential equations evoke n arbitrary constants. In most cases, we will be able to evaluate these constants if we know n initial values $y(x_0), y'(x_0), \ldots, y^{(n-1)}(x_0)$.

Initial Value Problem

Definition 3. By an **initial value problem** for an nth-order differential equation

$$F\left(x, y, \frac{dy}{dx}, \ldots, \frac{d^ny}{dx^n}\right) = 0,$$

we mean: Find a solution to the differential equation on an interval I that satisfies at x_0 the n initial conditions

$$\begin{aligned} y(x_0) &= y_0, \\ \frac{dy}{dx}(x_0) &= y_1, \\ &\vdots \\ \frac{d^{n-1}y}{dx^{n-1}}(x_0) &= y_{n-1}, \end{aligned}$$

where $x_0 \in I$ and $y_0, y_1, \ldots, y_{n-1}$ are given constants.

In the case of a first-order equation, the initial conditions reduce to the single requirement

$$y(x_0) = y_0,$$

and in the case of a second-order equation, the initial conditions have the form

$$y(x_0) = y_0, \qquad \frac{dy}{dx}(x_0) = y_1.$$

The terminology *initial conditions* comes from mechanics, where the independent variable x represents *time* and is customarily symbolized as t. Then if t_0 is the starting time, $y(t_0) = y_0$ represents the initial location of an object and $y'(t_0)$ gives its initial velocity.

Example 6 Show that $\phi(x) = \sin x - \cos x$ is a solution to the initial value problem

$$\textbf{(10)} \qquad \frac{d^2y}{dx^2} + y = 0; \qquad y(0) = -1, \qquad \frac{dy}{dx}(0) = 1.$$

Solution Observe that $\phi(x) = \sin x - \cos x$, $d\phi/dx = \cos x + \sin x$, and $d^2\phi/dx^2 = -\sin x + \cos x$ are all defined on $(-\infty, \infty)$. Substituting into the differential equation gives

$$(-\sin x + \cos x) + (\sin x - \cos x) = 0,$$

which holds for all $x \in (-\infty, \infty)$. Hence, $\phi(x)$ is a solution to the differential equation in (10) on $(-\infty, \infty)$. When we check the initial conditions, we find

$$\begin{aligned} \phi(0) &= \sin 0 - \cos 0 = -1, \\ \frac{d\phi}{dx}(0) &= \cos 0 + \sin 0 = 1, \end{aligned}$$

which meets the requirements of (10). Therefore, $\phi(x)$ is a solution to the given initial value problem. ◆

Example 7 As shown in Example 2, the function $\phi(x) = c_1e^{-x} + c_2e^{2x}$ is a solution to

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that the initial conditions

$$y(0) = 2 \quad \text{and} \quad \frac{dy}{dx}(0) = -3$$

are satisfied.

Solution To determine the constants c_1 and c_2, we first compute $d\phi/dx$ to get $d\phi/dx = -c_1e^{-x} + 2c_2e^{2x}$. Substituting in our initial conditions gives the following system of equations:

$$\begin{cases} \phi(0) = c_1e^0 + c_2e^0 = 2 \ , \\ \dfrac{d\phi}{dx}(0) = -c_1e^0 + 2c_2e^0 = -3 \ , \end{cases} \quad \text{or} \quad \begin{cases} c_1 + c_2 = 2 \ , \\ -c_1 + 2c_2 = -3 \ . \end{cases}$$

Adding the last two equations yields $3c_2 = -1$, so $c_2 = -1/3$. Since $c_1 + c_2 = 2$, we find $c_1 = 7/3$. Hence, the solution to the initial value problem is $\phi(x) = (7/3)e^{-x} - (1/3)e^{2x}$. ◆

We now state an existence and uniqueness theorem for first-order initial value problems. We presume the differential equation has been cast into the format

$$\frac{dy}{dx} = f(x, y) \ .$$

Of course, the right-hand side, $f(x, y)$, must be well defined at the starting value x_0 for x and at the stipulated initial value $y_0 = y(x_0)$ for y. The hypotheses of the theorem, moreover, require *continuity* of both f and $\partial f/\partial y$ for x in some interval $a < x < b$ containing x_0, and for y in some interval $c < y < d$ containing y_0. Notice that the set of points in the xy-plane that satisfy $a < x < b$ and $c < y < d$ constitutes a *rectangle*. Figure 1.5 on page 12 depicts this "rectangle of continuity" with the initial point (x_0, y_0) in its interior and a sketch of a portion of the solution curve contained therein.

Existence and Uniqueness of Solution

Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ .$$

If f and $\partial f/\partial y$ are continuous functions in some rectangle

$$R = \{(x, y): a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.†

†We remark that the continuity of f alone in such a rectangle is enough to guarantee the existence of a solution to the initial value problem in some open interval containing x_0, but uniqueness may not hold (see Example 9).

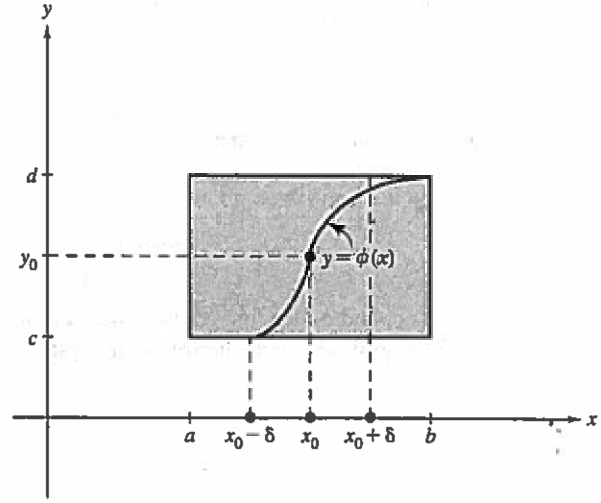

Figure 1.5 Layout for the existence–uniqueness theorem

The preceding theorem tells us two things. First, when an equation satisfies the hypotheses of Theorem 1, we are assured that a solution to the initial value problem exists. Naturally, it is desirable to know whether the equation we are trying to solve actually has a solution before we spend too much time trying to solve it. Second, when the hypotheses are satisfied, there is a **unique** solution to the initial value problem. This uniqueness tells us that if we can find a solution, then it is the *only* solution for the initial value problem. Graphically, the theorem says that there is only one solution curve that passes through the point (x_0, y_0). In other words, for this first-order equation, two solutions cannot cross anywhere in the rectangle. Notice that the existence and uniqueness of the solution holds only in *some* neighborhood $(x_0 - \delta, x_0 + \delta)$. Unfortunately, the theorem does not tell us the span (2δ) of this neighborhood (merely that it is not zero). Problem 18 elaborates on this feature.

Problem 19 gives an example of an equation with no solution. Problem 29 displays an initial value problem for which the solution is not unique. Of course, the hypotheses of Theorem 1 are not met for these cases.

When initial value problems are used to model physical phenomena, many practitioners tacitly presume the conclusions of Theorem 1 to be valid. Indeed, for the initial value problem to be a reasonable model, we certainly expect it to have a solution, since physically "something does happen." Moreover, the solution should be unique in those cases when repetition of the experiment under identical conditions yields the same results.[†]

The proof of Theorem 1 involves converting the initial value problem into an integral equation and then using Picard's method to generate a sequence of successive approximations that converge to the solution. The conversion to an integral equation and Picard's method are discussed in Project A at the end of this chapter. A detailed discussion and proof of the theorem are given in Chapter 13.[‡]

[†]At least this is the case when we are considering a deterministic model, as opposed to a probabilistic model.

[‡]All references to Chapters 11–13 refer to the expanded text, *Fundamentals of Differential Equations and Boundary Value Problems*, 7th ed.

Example 8 For the initial value problem

$$(11) \qquad 3\frac{dy}{dx} = x^2 - xy^3 , \qquad y(1) = 6 ,$$

does Theorem 1 imply the existence of a unique solution?

Solution Dividing by 3 to conform to the statement of the theorem, we identify $f(x, y)$ as $(x^2 - xy^3)/3$ and $\partial f/\partial y$ as $-xy^2$. Both of these functions are continuous in any rectangle containing the point $(1, 6)$, so the hypotheses of Theorem 1 are satisfied. It then follows from the theorem that the initial value problem (11) has a unique solution in an interval about $x = 1$ of the form $(1 - \delta, 1 + \delta)$, where δ is some positive number. ◆

Example 9 For the initial value problem

$$(12) \qquad \frac{dy}{dx} = 3y^{2/3} , \qquad y(2) = 0 ,$$

does Theorem 1 imply the existence of a unique solution?

Solution Here $f(x, y) = 3y^{2/3}$ and $\partial f/\partial y = 2y^{-1/3}$. Unfortunately $\partial f/\partial y$ is not continuous or even defined when $y = 0$. Consequently, there is no rectangle containing $(2, 0)$ in which both f and $\partial f/\partial y$ are continuous. Because the hypotheses of Theorem 1 do not hold, we cannot use Theorem 1 to determine whether the initial value problem does or does not have a unique solution. It turns out that this initial value problem has more than one solution. We refer you to Problem 29 and Project G of Chapter 2 for the details. ◆

In Example 9 suppose the initial condition is changed to $y(2) = 1$. Then, since f and $\partial f/\partial y$ are continuous in any rectangle that contains the point $(2, 1)$ but does not intersect the x-axis—say, $R = \{(x, y): 0 < x < 10, 0 < y < 5\}$—it follows from Theorem 1 that this *new* initial value problem has a unique solution in some interval about $x = 2$.

1.2 EXERCISES

1. (a) Show that $\phi(x) = x^2$ is an explicit solution to
$$x\frac{dy}{dx} = 2y$$
on the interval $(-\infty, \infty)$.
(b) Show that $\phi(x) = e^x - x$ is an explicit solution to
$$\frac{dy}{dx} + y^2 = e^{2x} + (1 - 2x)e^x + x^2 - 1$$
on the interval $(-\infty, \infty)$.
(c) Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to $x^2 d^2y/dx^2 = 2y$ on the interval $(0, \infty)$.

2. (a) Show that $y^2 + x - 3 = 0$ is an implicit solution to $dy/dx = -1/(2y)$ on the interval $(-\infty, 3)$.
(b) Show that $xy^3 - xy^3 \sin x = 1$ is an implicit solution to
$$\frac{dy}{dx} = \frac{(x \cos x + \sin x - 1)y}{3(x - x \sin x)}$$
on the interval $(0, \pi/2)$.

In Problems 3–8, determine whether the given function is a solution to the given differential equation.

3. $y = \sin x + x^2$, $\qquad \dfrac{d^2y}{dx^2} + y = x^2 + 2$

4. $x = 2 \cos t - 3 \sin t$, $\qquad x'' + x = 0$

5. $\theta = 2e^{3t} - e^{2t}$, $\qquad \dfrac{d^2\theta}{dt^2} - \theta\dfrac{d\theta}{dt} + 3\theta = -2e^{2t}$

6. $x = \cos 2t$, $\quad \dfrac{dx}{dt} + tx = \sin 2t$

7. $y = e^{2x} - 3e^{-x}$, $\quad \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 2y = 0$

8. $y = 3\sin 2x + e^{-x}$, $\quad y'' + 4y = 5e^{-x}$

In Problems 9–13, determine whether the given relation is an implicit solution to the given differential equation. Assume that the relationship does define y implicitly as a function of x and use implicit differentiation.

9. $x^2 + y^2 = 4$, $\quad \dfrac{dy}{dx} = \dfrac{x}{y}$

10. $y - \ln y = x^2 + 1$, $\quad \dfrac{dy}{dx} = \dfrac{2xy}{y-1}$

11. $e^{xy} + y = x - 1$, $\quad \dfrac{dy}{dx} = \dfrac{e^{-xy} - y}{e^{-xy} + x}$

12. $x^2 - \sin(x+y) = 1$, $\quad \dfrac{dy}{dx} = 2x\sec(x+y) - 1$

13. $\sin y + xy - x^3 = 2$,
$$y'' = \frac{6xy' + (y')^3 \sin y - 2(y')^2}{3x^2 - y}$$

14. Show that $\phi(x) = c_1 \sin x + c_2 \cos x$ is a solution to $d^2y/dx^2 + y = 0$ for any choice of the constants c_1 and c_2. Thus, $c_1 \sin x + c_2 \cos x$ is a two-parameter family of solutions to the differential equation.

15. Verify that $\phi(x) = 2/(1 - ce^x)$, where c is an arbitrary constant, is a one-parameter family of solutions to
$$\frac{dy}{dx} = \frac{y(y-2)}{2}.$$
Graph the solution curves corresponding to $c = 0, \pm 1, \pm 2$ using the same coordinate axes.

16. Verify that $x^2 + cy^2 = 1$, where c is an arbitrary nonzero constant, is a one-parameter family of implicit solutions to
$$\frac{dy}{dx} = \frac{xy}{x^2 - 1}$$
and graph several of the solution curves using the same coordinate axes.

17. Show that $\phi(x) = Ce^{3x} + 1$ is a solution to $dy/dx - 3y = -3$ for any choice of the constant C. Thus, $Ce^{3x} + 1$ is a one-parameter family of solutions to the differential equation. Graph several of the solution curves using the same coordinate axes.

18. Let $c > 0$. Show that the function $\phi(x) = (c^2 - x^2)^{-1}$ is a solution to the initial value problem $dy/dx = 2xy^2$, $y(0) = 1/c^2$, on the interval $-c < x < c$. Note that this solution becomes unbounded as x approaches $\pm c$. Thus, the solution exists on the interval $(-\delta, \delta)$ with $\delta = c$, but not for larger δ. This illustrates that in Theorem 1 the existence interval can be quite small (if c is small) or quite large (if c is large). Notice also that there is no clue from the equation $dy/dx = 2xy^2$ itself, or from the initial value, that the solution will "blow up" at $x = \pm c$.

19. Show that the equation $(dy/dx)^2 + y^2 + 4 = 0$ has no (real-valued) solution.

20. Determine for which values of m the function $\phi(x) = e^{mx}$ is a solution to the given equation.

(a) $\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + 5y = 0$

(b) $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$

21. Determine for which values of m the function $\phi(x) = x^m$ is a solution to the given equation.

(a) $3x^2\dfrac{d^2y}{dx^2} + 11x\dfrac{dy}{dx} - 3y = 0$

(b) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 5y = 0$

22. Verify that the function $\phi(x) = c_1e^x + c_2e^{-2x}$ is a solution to the linear equation
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$
for any choice of the constants c_1 and c_2. Determine c_1 and c_2 so that each of the following initial conditions is satisfied.

(a) $y(0) = 2$, $\quad y'(0) = 1$

(b) $y(1) = 1$, $\quad y'(1) = 0$

In Problems 23–28, determine whether Theorem 1 implies that the given initial value problem has a unique solution.

23. $\dfrac{dy}{dx} = y^4 - x^4$, $\quad y(0) = 7$

24. $\dfrac{dy}{dt} - ty = \sin^2 t$, $\quad y(\pi) = 5$

25. $3x\dfrac{dx}{dt} + 4t = 0$, $\quad x(2) = -\pi$

26. $\dfrac{dx}{dt} + \cos x = \sin t$, $\quad x(\pi) = 0$

27. $y\dfrac{dy}{dx} = x$, $\quad y(1) = 0$

28. $\dfrac{dy}{dx} = 3x - \sqrt[3]{y-1}$, $\quad y(2) = 1$

29. (a) For the initial value problem (12) of Example 9, show that $\phi_1(x) \equiv 0$ and $\phi_2(x) = (x-2)^3$ are solutions. Hence, this initial value problem has multiple solutions. (See also Project G in Chapter 2.)

(b) Does the initial value problem $y' = 3y^{2/3}$, $y(0) = 10^{-7}$, have a unique solution in a neighborhood of $x = 0$?

30. Implicit Function Theorem. *Let $G(x, y)$ have continuous first partial derivatives in the rectangle $R = \{(x, y): a < x < b, c < y < d\}$ containing the point (x_0, y_0). If $G(x_0, y_0) = 0$ and the partial derivative $G_y(x_0, y_0) \neq 0$, then there exists a differentiable function $y = \phi(x)$, defined in some interval $I = (x_0 - \delta, x_0 + \delta)$, that satisfies $G(x, \phi(x)) = 0$ for all $x \in I$.*

The implicit function theorem gives conditions under which the relationship $G(x, y) = 0$ defines y implicitly as a function of x. Use the implicit function theorem to show that the relationship $x + y + e^{xy} = 0$, given in Example 4, defines y implicitly as a function of x near the point $(0, -1)$.

31. Consider the equation of Example 5,

$$y\frac{dy}{dx} - 4x = 0 \,. \tag{13}$$

(a) Does Theorem 1 imply the existence of a unique solution to (13) that satisfies $y(x_0) = 0$?

(b) Show that when $x_0 \neq 0$, equation (13) can't possibly have a solution in a neighborhood of $x = x_0$ that satisfies $y(x_0) = 0$.

(c) Show that there are two distinct solutions to (13) satisfying $y(0) = 0$ (see Figure 1.4 on page 9).

1.3 Direction Fields

The existence and uniqueness theorem discussed in Section 1.2 certainly has great value, but it stops short of telling us anything about the *nature* of the solution to a differential equation. For practical reasons we may need to know the value of the solution at a certain point, or the intervals where the solution is increasing, or the points where the solution attains a maximum value. Certainly, knowing an explicit representation (a formula) for the solution would be a considerable help in answering these questions. However, for many of the differential equations that we are likely to encounter in real-world applications, it will be impossible to find such a formula. Moreover, even if we are lucky enough to obtain an implicit solution, using this relationship to determine an explicit form may be difficult. Thus, we must rely on other methods to analyze or approximate the solution.

One technique that is useful in visualizing (graphing) the solutions to a first-order differential equation is to sketch the direction field for the equation. To describe this method, we need to make a general observation. Namely, a first-order equation

$$\frac{dy}{dx} = f(x, y)$$

specifies a slope at each point in the xy-plane where f is defined. In other words, it gives the direction that a graph of a solution to the equation must have at each point. Consider, for example, the equation

$$\frac{dy}{dx} = x^2 - y \,. \tag{1}$$

The graph of a solution to (1) that passes through the point $(-2, 1)$ must have slope $(-2)^2 - 1 = 3$ at that point, and a solution through $(-1, 1)$ has zero slope at that point.

A plot of short line segments drawn at various points in the xy-plane showing the slope of the solution curve there is called a **direction field** for the differential equation. Because the direction field gives the "flow of solutions," it facilitates the drawing of any particular solution (such as the solution to an initial value problem). In Figure 1.6(a) on page 16 we have sketched the direction field for equation (1) and in Figure 1.6(b) we have drawn several solution curves in color.