Non-book problems

1. Verify that for any nonzero constant b, the function $f(x) = \frac{1}{b}\cosh(bx)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - b\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0.$$

(Recall that the function "cosh" is defined by $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

2. Verify that on the interval $-2 < x < 2$, the two continuous functions $y(x) = \sqrt{4 - x^2}$ and $y(x) = -\sqrt{4 - x^2}$, obtainable from the implicit solution $x^2 + y^2 = 4$ of the differential equation $x + y\ dy/dx = 0$, are (explicit) solutions of this differential equation.

3. Find the general solution of

$$\frac{dy}{dx} = \frac{x \sin x}{\ln y} \quad .$$

4. Find the general solution of

$$\frac{dy}{dx} = \frac{\tan^{-1} x}{y e^{2y}} \quad .$$

(*Notational reminder*: "$\tan^{-1}$" denotes the inverse-tangent function, also called arctangent, and also written "arctan". It does *not* denote the reciprocal of the tangent function, which is the cotangent function "cot".)

5. Consider the initial-value problem

$$\frac{dx}{dt} + x^2 = x, \quad x(0) = x_0 \tag{1}$$

(This is the same DE as in exercise 13 of Section 2.2, which you should do prior to starting this exercise.)

For each of the following values of x_0, find both the solution of the IVP and the domain of the solution. The answers to "What is the domain of the solution?" are given below, but *they are not obvious.* Do *not* expect to be able to find a quick way for guessing what the answers will be based on x_0. To get these answers you will have to *solve the IVP first*—you should get an explicit formula for $x(t)$—and then, from your formula, figure out the domain of the solution (remembering that only *intervals* are allowed as solutions of ODEs, even if the formula you write down has a larger domain).

(a) $x_0 = \frac{1}{2}$
Answer for domain: $(-\infty, \infty)$ *(i.e.* $-\infty < t < \infty$*)*

(b) $x_0 = 2$
Answer for domain: $(-\ln 2, \infty)$ *(i.e.* $-\ln 2 < t < \infty$*)*

(c) $x_0 = -2$
Answer for domain: $(-\infty, \ln(\frac{3}{2}))$ *(i.e.* $-\infty < t < \ln(\frac{3}{2})$*)*

(d) $x_0 = 0$
Answer for domain: $(-\infty, \infty)$

(e) $x_0 = 1$
Answer for domain: $(-\infty, \infty)$

6. Let p be a function that is differentiable on the whole real line, and consider the separable differential equation

$$\frac{dy}{dx} = p(y). \tag{2}$$

(Here, the function $g(x)$ that you're used to seeing is just the constant function 1.) Note that if we rename the variables x, t in the previous exercise to y, x respectively, the ODE in that exercise can be rewritten in this form, with $p(y) = y - y^2$.

(a) Show that the family of all solutions of (2) is *translation-invariant* in the following sense: if $y = \phi(x)$ is a solution on an interval $a < x < b$, and k is any constant, then $y = \phi(x - k)$ is a solution on the interval $a + k < x < b + k$. (Said another way: horizontally translating the graph of a solution by any amount, you get the graph of another solution.)

(b) Using the Fundamental Existence/Uniqueness Theorem for First-Order Initial-Value Problems (Theorem 1 on p. 12 in the textbook), show that for every point $(x_0, y_0) \in \mathbf{R}^2$, the initial-value problem

$$\frac{dy}{dx} = p(y), \quad y(x_0) = y_0, \tag{3}$$

has a unique solution on some open interval containing x_0.

(c) Assume that there are numbers $c < d$ such that $p(c) = p(d) = 0$. Use the "Uniqueness" part of the Fundamental Existence/Uniqueness Theorem to show all of the following. (Once you see how to do any one of these, the other two should be easy.)

- (i) If $y_0 > d$, and ϕ is a solution of (3) defined on an open interval I_{x_0} containing x_0, then $\phi(x) > d$ for all $x \in I_{x_0}$. (Note: you are not allowed to assume that I_{x_0} is a *small* interval; you have to show that what's stated is true no matter how large I_{x_0} is. I_{x_0} could even be the whole real line.)

- (ii) If $y_0 < c$, then the solution ϕ of (3) satisfies $\phi(x) < c$ for all $x \in I_{x_0}$. (Same note as above applies.)

- (iii) If $c < y_0 < d$, then the solution ϕ of (3) satisfies $\phi(x) > d$ for all $x \in I_{x_0}$. (Same note as above applies.)

(d) Check that the solutions you found in Exercise 5abc above are consistent with what you showed (or were told to show) in part (c) of the current exercise.

7. Solve the following differential equations.
(a) $\frac{du}{dt} + \frac{2}{t}u = e^t, \quad t < 0.$

(b) $\frac{dy}{dx} - (\tan x)y = \sec x \ln x, \quad 0 < x < \pi/2.$

(c) $x^2 \frac{dy}{dx} - 3xy = x^6 \tan^{-1} x.$

8. Show that if F_1 and F_2 are differentiable functions on an open rectangle R in the xy plane, and $dF_2 = dF_1$ throughout R, then F_1 and F_2 differ by a constant (i.e. there is a constant C such that $F_2(x, y) = F_1(x, y) + C$ for all $(x, y) \in R$).

9. **Passing the "Exactness Test" not sufficient for exactness on domain with a hole.** As discussed in class and in the book, if M and N are continuously differentiable (i.e. have continuous first partial derivatives) on an open rectangle R in the xy plane, and $M_y = N_x$ throughout R, then $Mdx + Ndy$ is exact on R. A rectangle is an example of what mathematicians call a *simply connected* region: a region with "no holes". (The intuitive notion of "no holes" can be given a precise definition, but not in MAP 2302.) It can be shown that on any simply connected region R, not just rectangles, if M and N are continuously differentiable, then $Mdx + Ndy$ is exact on R if and only if $M_y = N_x$ throughout R.

If R is not simply connected, then "$M_y = N_x$" is still a *necessary* condition for exactness on R, but not a *sufficient* condition: there are always differentials that satisfy $M_y = N_x$, but that are not exact. You will construct an example in this exercise. The non-simply-connected region we will use is

$$R = \{(x, y) \in \mathbf{R}^2 \mid (x, y) \neq (0, 0)\}, \tag{4}$$

i.e. $\mathbf{R}^2$ with the origin removed. This region has a "hole" at the origin. On R, define

$$M(x, y) = \frac{-y}{x^2 + y^2}\ , \quad N(x, y) = \frac{x}{x^2 + y^2}\ . \tag{5}$$

For the rest of this exercise, "R" always means the region in (4), and "M" and "N" always mean the functions in (5).

(a) Show that M and N are continuously differentiable on R and that $M_y = N_x$ throughout R.

(b) Show that on the set $\{(x, y) \in \mathbf{R}^2 \mid x \text{ and } y \text{ are both nonzero}\}$ (i.e. $\mathbf{R}^2$ with the coordinate axes removed),

$$M(x, y)dx + N(x, y)dy = d(\tan^{-1}(\frac{y}{x})) = d(-\tan^{-1}(\frac{x}{y})).$$

(c) Define four functions as follows, with the indicated domains.

$$\begin{aligned}
F_{\text{right}}(x, y) &= \tan^{-1}(\frac{y}{x}), \quad x > 0. \\
F_{\text{upper}}(x, y) &= -\tan^{-1}(\frac{x}{y}) + \frac{\pi}{2}, \quad y > 0. \\
F_{\text{left}}(x, y) &= \tan^{-1}(\frac{y}{x}) + \pi, \quad x < 0. \\
F_{\text{lower}}(x, y) &= -\tan^{-1}(\frac{x}{y}) + \frac{3\pi}{2}, \quad y < 0.
\end{aligned}$$

Show that the following four identities hold:

$$\begin{aligned}
F_{\text{upper}}(x,y) &= F_{\text{right}}(x,y) \;\; \text{throughout open quadrant I.} \\
F_{\text{left}}(x,y) &= F_{\text{upper}}(x,y) \;\; \text{throughout open quadrant II.} \\
F_{\text{lower}}(x,y) &= F_{\text{left}}(x,y) \;\; \text{throughout open quadrant III.} \\
F_{\text{right}}(x,y) &= F_{\text{lower}}(x,y) + 2\pi \;\; \text{throughout open quadrant IV.}
\end{aligned}$$

Quadrants I–IV are the usual quadrants of the xy plane, and "open quadrant" means "quadrant with the coordinate axes removed".

(d) Use the result of exercise 8 (of these non-book problems) to show the following:

- F_{upper} is the *only* continuously differentiable function defined on the entire open upper half-plane $\{(x,y) \in \mathbf{R}^2 : y > 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{right} on open quadrant I.
- F_{left} is the *only* continuously differentiable function defined on the entire open left half-plane $\{(x,y) \in \mathbf{R}^2 : x < 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{upper} on open quadrant II.
- F_{lower} is the *only* continuously differentiable function defined on the entire open lower half-plane $\{(x,y) \in \mathbf{R}^2 : y < 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{left} on open quadrant III.

(e) Show that because the identities in part (c) hold, the following definition of a function F on the domain

$$\{(x,y) \in \mathbf{R}^2 : (x,y) \neq (a,0) \text{ for any } a \geq 0\} \tag{6}$$

(i.e. $\mathbf{R}^2$ with the origin and positive x-axis removed) is unambiguous, even though within each open quadrant the definition gives two different formulas for F:

$$\begin{aligned}
F(x,y) &= F_{\text{right}}(x,y) \quad \text{in open quadrant I.} \\
F(x,y) &= F_{\text{upper}}(x,y) \quad \text{if } y > 0, \;\; \text{i.e. for } (x,y) \text{ in the open upper half-plane.} \\
F(x,y) &= F_{\text{left}}(x,y) \quad \text{if } x < 0, \;\; \text{i.e. for } (x,y) \text{ in the open left half-plane.} \\
F(x,y) &= F_{\text{lower}}(x,y) \quad \text{if } y < 0, \;\; \text{i.e. for } (x,y) \text{ in the open lower half-plane.} \\
F(x,y) &= F_{\text{right}}(x,y) + 2\pi \quad \text{in open quadrant IV.}
\end{aligned}$$

This function has a simple geometric interpretation: $F(x,y)$ is the polar coordinate $\theta \in (0, 2\pi)$ of the point (x,y).

(f) Use part (d) to show that F is the *only* differentiable function defined on the domain (6) whose differential is $M\,dx + N\,dy$ on this domain and that coincides with F_{right} on open quadrant I.

(g) Show that for all $x_0 > 0$, $\lim_{y\to 0+} F(x_0, y) = 0$, while $\lim_{y\to 0-} F(x_0, y) = 2\pi$.

(h) Use part (g) to show that there is *no* continuous function defined on the whole domain R (see (4)) that coincides with F on the domain (6). Then, combine this fact with part (f) to show that there is *no* continuously differentiable function on R whose differential is $M\,dx + N\,dy$ on this domain and that coincides with F_{upper} on open quadrant I.

(i) Use exercise 8 to show that if G is any differentiable function defined on open quadrant I for which $dG = Mdx + Ndy$, then, on open quadrant I, G differs from F_{upper} only by an additive constant.

(j) Use parts (h) and (i) to show that there is *no* differentiable function H defined on all of R for which $dH = Mdx + Ndy$. Thus, $Mdx + Ndy$ is not exact on R, despite satisfying $M_y = N_x$ at every point of R.

Fact: It is accepted practice to write "$d\theta$" for the differential $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on R, even though there is *no* differentiable function θ defined on all of R whose differential is $d\theta$!

10. Show that if L_1 and L_2 are linear operators, then $L_1 + L_2$ is a linear operator.

11. Show by induction on n that if an operator L is linear, then for all $n \geq 1$, all constants $c_1, c_2, \ldots, c_n$, and all functions $f_1, f_2, \ldots f_n$,

$$L[c_1 f_1 + c_2 f_2 + \ldots + c_n f_n] = c_1 L[f_1] + c_2 L[f_2] + \ldots + c_n L[f_n].$$

12. Show that the following distributive law holds for *linear* operators: if L_1, L_2, R_1, and R_2 are linear operators, then

$$(L_1 + L_2)(R_1 + R_2) = L_1R_1 + L_1R_2 + L_2R_1 + L_2R_2.$$

Note: Because linear operators don't commute with each other in general (see the next exercise), the formula above is valid *only* with the L's in front of the R's on the right-hand side of the equation.

13. General (i.e. not constant-coefficient) linear differential operators L_1, L_2 do not commute with each other: $L_1L_2 \neq L_2L_1$. Part (a) of this problem verifies this statement by giving examples of operators that do not commute. It is still true that *some* linear differential operators commute with each other; you will see examples of this in parts (b) and (c).

(a) Let L_1 be the operator "multiplication by p", where p is a *non-constant* function, let $L_2 = D$ (the first-derivative operator), and let L_3 be defined by $L_3[f](t) = tf'(t)$. (When it is agreed in advance that the letter t will be used for the independent variable, we may write the definition of L_3 as "$L_3 = tD$". The operator L_1 may not look like a *differential* operator to you, since there are no derivatives involved. It happens still to be called a differential operator, but its order is zero.) Show that $L_1L_2 \neq L_2L_1$ and that $L_3L_2 \neq L_2L_3$.

Note: to show, for example, that $L_1L_2 \neq L_2L_1$, compute what both L_1L_2 and L_2L_1 do to a general differentiable function f, and see that the result is different for L_2L_1 from what it was for L_1L_2.

(b) Let L_1 and L_2 be as in part (a), but this time assume that p is a *constant* function. Show that in this case $L_1L_2 = L_2L_1$. (I.e. show that $(L_1L_2)[f] = (L_2L_1)[f]$ for all twice-differentiable functions f.)

(c) Let a and b be constants, and let L_1 and L_2 be the linear differential operators $D + a$ and $D + b$, respectively. Using part (b) plus Exercise 12 above, show that $L_1L_2 = L_2L_1$.

What you have shown in parts (a), (b) and (c) are special cases of a more general principle: all *constant-coefficient* linear differential operators $a_nD^n + a_{n-1}D^{n-1} + \ldots + a_1D + a_0$ commute with each other (here the a's are constants), but *non-constant-coefficient* linear differential operators $p_nD^n + p_{n-1}D^{n-1} + \ldots + p_1D + p_0$ (here the p's are functions at least one of which is non-constant) usually don't commute with other linear differential operators (whether or not the other operators are constant-coefficient).

14. The method of the textbook's Examples 2–3 on p. 209, and in exercises 4.7/23–24, can be adapted to give solutions to several cases of Cauchy-Euler (pronounced "Co-she Oiler") equations not covered in the book's exercises. In this problem we consider these other cases.

(a) Fix numbers a, b, c (with $a \neq 0$) and consider the second-order homogeneous Cauchy-Euler equation

$$at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0. \tag{7}$$

If we consider this equation on the interval $\{t < 0\}$, the substitution $t = e^x$ cannot be used (why not?). However, using the Chain Rule and the substitution $t = -u$, show that (7) for $t < 0$ is equivalent to the equation

$$au^2\frac{d^2z}{du^2} + bu\frac{dz}{du} + cz = 0 \tag{8}$$

for $u > 0$, where $z(u) = y(t) = z(-t)$. Except for the names of the variables, equations (7) and (8) are the same. Use this to show that if $t \mapsto \phi(t)$ is a solution of $at^2y'' + bty' + cy = 0$ on the interval $\{t > 0\}$, then $t \mapsto \phi(-t)$ is a solution of the same DE on the interval $\{t < 0\}$, and vice-versa. Thus show that if $t \mapsto y_{\text{gen}}(t)$ is the general solution of (7) on the interval $\{t > 0\}$, then $t \mapsto y_{\text{gen}}(|t|)$ is the general solution of (7) on the interval $\{t < 0\}$.

(b) Using the method of 4.7/23–24, find the general solution $t \mapsto y(t)$ of

$$6t^2y'' + ty' + y = 0 \tag{9}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (9) on the interval $\{t < 0\}$.

(c) Using the methods of the book's 4.7/23–24, find the general solution $t \mapsto y(t)$ of

$$t^2y'' + 5ty' + 4y = 0 \tag{10}$$

on the interval $\{t > 0\}$. (Remember that your answer must be expressed purely in terms of t, not partly in terms of t and partly in terms of x.) Then, using part (a) above, find the general solution of (10) on the interval $\{t < 0\}$.