Non-book problems

1. Verify that for any nonzero constant b, the function $f(x) = \frac{1}{b}\cosh(bx)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - b\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0.$$

(Recall that the function "cosh" is defined by $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

2. Consider the equation

$$x^2 + y^2 = 4. \tag{1}$$

On the interval $-2 < x < 2$, there are two continuous functions of x determined by (1). These may be expressed by the equations $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. Show directly, without implicit differentiation, that each of these two equations is an explicit solution of the differential equation $x + y\ dy/dx = 0$. (Note that this is the DE obtained by implicitly differentiating (1) with respect to x, and then dividing by 2 just to simplify.)

3. Solve the equation

$$\frac{dy}{dx} = \frac{x \sin x}{\ln y} \quad .$$

4. Solve the equation

$$\frac{dy}{dx} = \frac{\tan^{-1} x}{ye^{2y}} \quad .$$

(*Notational reminder*: "$\tan^{-1}$" denotes the inverse-tangent function, also called arctangent, and also written "arctan". It does *not* denote the reciprocal of the tangent function, which is the cotangent function "cot".)

5. Let p be a function that is differentiable on the whole real line, and consider the separable differential equation

$$\frac{dy}{dx} = p(y). \tag{2}$$

(Here, the function $g(x)$ that you're used to seeing is just the constant function 1.)

(a) Show that the family of all solutions of (2) is *translation-invariant* in the following sense: if $y = \phi(x)$ is a solution on an interval $a < x < b$, and k is any constant, then $y = \phi(x - k)$ is a solution on the interval $a + k < x < b + k$. (Said another way: horizontally translating the graph of a solution by any amount, you get the graph of another solution.)

(b) Using the Fundamental Existence/Uniqueness Theorem for First-Order Initial-Value Problems (Theorem 1 on p. 11 in the textbook), show that for every point $(x_0, y_0) \in \mathbf{R}^2$, the initial-value problem

$$\frac{dy}{dx} = p(y), \quad y(x_0) = y_0, \tag{3}$$

has a unique solution on some open interval containing x_0.

(c) Assume that there are numbers $c < d$ such that $p(c) = p(d) = 0$. Use the "Uniqueness" part of the Fundamental Existence/Uniqueness Theorem to show all of the following. (Once you see how to do any one of these, the other two should be easy.)

- (i) If $y_0 > d$, and ϕ is a solution of (3) defined on an open interval I_{x_0} containing x_0, then $\phi(x) > d$ for all $x \in I_{x_0}$. (Note: you are not allowed to assume that I_{x_0} is a *small* interval; you have to show that what's stated is true no matter how large I_{x_0} is. I_{x_0} could even be the whole real line.)
- (ii) If $y_0 < c$, then the solution ϕ of (3) satisfies $\phi(x) < c$ for all $x \in I_{x_0}$. (Same note as above applies.)
- (iii) If $c < y_0 < d$, then the solution ϕ of (3) satisfies $\phi(x) > d$ for all $x \in I_{x_0}$. (Same note as above applies.)

6. Solve the differential equation $\frac{dy}{dx} = 2xy(1-y^2)$.

7. Solve the following differential equations.
(a) $\frac{du}{dt} + \frac{2}{t}u = e^t, \quad t < 0.$
(b) $\frac{dy}{dx} - (\tan x)y = \sec x \ln x, \quad 0 < x < \pi/2.$
(c) $x^2\frac{dy}{dx} - 3xy = x^6 \tan^{-1} x.$

8. Show that if F_1 and F_2 are differentiable functions on an open rectangle R in the xy plane, and $dF_2 = dF_1$ throughout R, then F_1 and F_2 differ by a constant (i.e. there is a constant C such that $F_2(x, y) = F_1(x, y) + C$ for all $(x, y) \in R$).

9. **Passing the "Exactness Test" not sufficient for exactness on domain with a hole.** As discussed in class and in the book, if M and N are continuously differentiable (i.e. have continuous first partial derivatives) on an open rectangle R in the xy plane, and $M_y = N_x$ throughout R, then $Mdx + Ndy$ is exact on R. A rectangle is an example of what mathematicians call a *simply connected* region: a region with "no holes". (The intuitive notion of "no holes" can be given a precise definition, but not in MAP 2302.) It can be shown that on any simply connected region R, not just rectangles, if M and N are continuously differentiable, then $Mdx + Ndy$ is exact on R if and only if $M_y = N_x$ throughout R.

If R is not simply connected, then "$M_y = N_x$" is still a *necessary* condition for exactness on R, but not a *sufficient* condition: there are always differentials that satisfy $M_y = N_x$, but that are not exact. You will construct an example in this exercise. The non-simply-connected region we will use is

$$R = \{(x,y) \in \mathbf{R}^2 \mid (x,y) \neq (0,0)\}, \tag{4}$$

i.e. $\mathbf{R}^2$ with the origin removed. This region has a "hole" at the origin. On R, define

$$M(x,y) = \frac{-y}{x^2+y^2}\ , \quad N(x,y) = \frac{x}{x^2+y^2}\ . \tag{5}$$

For the rest of this exercise, "R" always means the region in (4), and "M" and "N" always mean the functions in (5).

(a) Show that M and N are continuously differentiable on R and that $M_y = N_x$ throughout R.

(b) Show that on the set $\{(x,y) \in \mathbf{R}^2 \mid x \text{ and } y \text{ are both nonzero}\}$ (i.e. $\mathbf{R}^2$ with the coordinate axes removed),

$$M(x,y)dx + N(x,y)dy = d(\tan^{-1}(\frac{y}{x})) = d(-\tan^{-1}(\frac{x}{y})).$$

(c) Define four functions as follows, with the indicated domains.

$$\begin{aligned} F_{\text{right}}(x,y) &= \tan^{-1}(\frac{y}{x}), \quad x > 0. \\ F_{\text{upper}}(x,y) &= -\tan^{-1}(\frac{x}{y}) + \frac{\pi}{2}, \quad y > 0. \\ F_{\text{left}}(x,y) &= \tan^{-1}(\frac{y}{x}) + \pi, \quad x < 0. \\ F_{\text{lower}}(x,y) &= -\tan^{-1}(\frac{x}{y}) + \frac{3\pi}{2}, \quad y < 0. \end{aligned}$$

Show that the following four identities hold:

$$\begin{aligned} F_{\text{upper}}(x,y) &= F_{\text{right}}(x,y) \ \text{ throughout open quadrant I.} \\ F_{\text{left}}(x,y) &= F_{\text{upper}}(x,y) \ \text{ throughout open quadrant II.} \\ F_{\text{lower}}(x,y) &= F_{\text{left}}(x,y) \ \text{ throughout open quadrant III.} \\ F_{\text{right}}(x,y) &= F_{\text{lower}}(x,y) + 2\pi \ \text{ throughout open quadrant IV.} \end{aligned}$$

Quadrants I–IV are the usual quadrants of the xy plane, and "open quadrant" means "quadrant with the coordinate axes removed".

(d) Use the result of exercise 8 (of these non-book problems) to show the following:

- F_{upper} is the *only* continuously differentiable function defined on the entire open upper half-plane $\{(x,y) \in \mathbf{R}^2 : y > 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{right} on open quadrant I.

- F_{left} is the *only* continuously differentiable function defined on the entire open left half-plane $\{(x,y) \in \mathbf{R}^2 : x < 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{upper} on open quadrant II.

- F_{lower} is the *only* continuously differentiable function defined on the entire open lower half-plane $\{(x,y) \in \mathbf{R}^2 : y < 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{left} on open quadrant III.

(e) Show that because the identities in part (c) hold, the following definition of a function F on the domain

$$\{(x,y) \in \mathbf{R}^2 : (x,y) \neq (a,0) \text{ for any } a \geq 0\} \tag{6}$$

(i.e. $\mathbf{R}^2$ with the origin and positive x-axis removed) is unambiguous, even though within each open quadrant the definition gives two different formulas for F:

$$\begin{aligned}
F(x,y) &= F_{\text{right}}(x,y) \quad \text{in open quadrant I.}\\
F(x,y) &= F_{\text{upper}}(x,y) \quad \text{if } y > 0, \ \text{ i.e. for } (x,y) \text{ in the open upper half-plane.}\\
F(x,y) &= F_{\text{left}}(x,y) \quad \text{if } x < 0, \ \text{ i.e. for } (x,y) \text{ in the open left half-plane.}\\
F(x,y) &= F_{\text{lower}}(x,y) \quad \text{if } y < 0, \ \text{ i.e. for } (x,y) \text{ in the open lower half-plane.}\\
F(x,y) &= F_{\text{right}}(x,y) + 2\pi \quad \text{in open quadrant IV.}
\end{aligned}$$

This function has a simple geometric interpretation: $F(x,y)$ is the polar coordinate $\theta \in (0, 2\pi)$ of the point (x,y).

(f) Use part (d) to show that F is the *only* differentiable function defined on the domain (6) whose differential is $M\,dx + N\,dy$ on this domain and that coincides with F_{right} on open quadrant I.

(g) Show that for all $x_0 > 0$, $\lim_{y \to 0+} F(x_0, y) = 0$, while $\lim_{y \to 0-} F(x_0, y) = 2\pi$.

(h) Use part (g) to show that there is *no* continuous function defined on the whole domain R (see (4)) that coincides with F on the domain (6). Then, combine this fact with part (f) to show that there is *no* continuously differentiable function on R whose differential is $M\,dx + N\,dy$ on this domain and that coincides with F_{upper} on open quadrant I.

(i) Use exercise 9 to show that if G is any continuously differentiable function defined on open quadrant I for which $dG = Mdx + Ndy$, then, on open quadrant I, G differs from F_{upper} only by an additive constant.

(j) Use parts (h) and (i) to show that there is *no* differentiable function H defined on all of R for which $dH = Mdx + Ndy$. Thus, $Mdx + Ndy$ is not exact on R, despite satisfying $M_y = N_x$ at every point of R.

Fact: It is accepted practice to write "$d\theta$" for the differential $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on R, even though there is *no* differentiable function θ defined on all of R whose differential is $d\theta$!

10. The method in the textbook for solving Cauchy-Euler (pronounced "Co-she Oiler") equations on the domain-interval $(0,\infty)$ can be modified to handle the domain-interval $(-\infty,0)$. In this problem we consider such problems.

(a) Fix numbers a, b, c (with $a \neq 0$) and consider the second-order homogeneous Cauchy-Euler equation

$$at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0. \tag{7}$$

If we consider this equation on the interval $\{t < 0\}$, the substitution $t = e^x$ cannot be used (why not?). However, using the Chain Rule and the substitution $t = -u$, show that (7) for $t < 0$ is equivalent to the equation

$$au^2\frac{d^2z}{du^2} + bu\frac{dz}{du} + cz = 0 \tag{8}$$

for $u > 0$, where $z(u) = y(t) = z(-t)$. Except for the names of the variables, equations (7) and (8) are the same. Use this to show that if $t \mapsto \phi(t)$ is a solution of $at^2y'' + bty' + cy = 0$ on the interval $\{t > 0\}$, then $t \mapsto \phi(-t)$ is a solution of the same DE on the interval $\{t < 0\}$, and vice-versa.[1] Thus show that if $t \mapsto y_{\text{gen}}(t)$ is the general solution of (7) on the interval $\{t > 0\}$, then $t \mapsto y_{\text{gen}}(|t|)$ is the general solution of (7) on the interval $\{t < 0\}$, as well as on the interval $\{t > 0\}$.

(b) Find the general solution $t \mapsto y(t)$ of

$$6t^2y'' + ty' + y = 0 \tag{9}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (9) on the interval $\{t < 0\}$.

[1]The symbol "$\mapsto$" is read "maps to" or "goes to". It provides a way of describing a function without declaring a name for that function or for the independent variable. The function $t \mapsto t^2$ is the same as the function $x \mapsto x^2$, $u \mapsto u^2$, etc.; all of these are notations for the *squaring function*. Instead of "$t \mapsto \phi(-t)$", we could say, "The function ψ defined by $\psi(t) = \phi(-t)$," but unless we're going to *use* the name ψ for this function later, it's inconvenient to have to write all that. Note that the expression "$\psi(-t)$" is not a function, unless we specify a certain convention. (At the beginning of our treatment of linear constant-coefficient DEs, I stated that our in-class convention would be that unless otherwise specified, the independent variable would be t, and that I'd refer to function f by what the function spits out when t is fed in, i.e. the number $f(t)$. This convention allowed us to say things like "$\{e^{2t}, e^{3t}\}$ is a FSS of $y'' - 5y' + 6y = 0$." But without stating such a convention explicitly, something like "e^{2t}" is not a function—it's just an *expression*—and instead of $\{e^{2t}, e^{3t}\}$ for a FSS we would either have to write something like $\{t \mapsto e^{2t}, t \mapsto e^{3t}\}$ (equivalently, $\{u \mapsto e^{2u}, z \mapsto e^{3z}\}$, etc., for any name for the input of these functions), or something more long-winded like "$\{y_1, y_2\}$, where y_1 and y_2 are the functions defined by $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$.")

(c) Find the general solution $t \mapsto y(t)$ of

$$t^2 y'' + 5ty' + 4y = 0 \tag{10}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (10) on the interval $\{t < 0\}$. (Remember, in all these problems, that your answer must be expressed purely in terms of t, not partly in terms of t and partly in terms of x.)

(d) Find the general solution $t \mapsto y(t)$ of

$$t^2 y'' + 2ty' + y = 0 \tag{11}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (11) on the interval $\{t < 0\}$.