A terrible way to solve exact equations.

Suppose that the equation $M(x,y)dx+N(x,y)dy=0$ is exact (i.e. $\partial M/\partial y=\partial N/\partial x$) on some rectangle R. In the book, and in class, we discussed a method for finding a function F of (x,y) such that $dF=M(x,y)dx+N(x,y)dy$ (i.e. for which $\frac{\partial F}{\partial x}=M$ and $\frac{\partial F}{\partial y}=N$) and leading us to a family of implicit solutions of the form $F(x,y)=\text{constant}$. Here is a **wrong** method for trying to find such an F. Make sure it's **not** the method you're using.

Step 1. Integrate M with respect to x, omitting the "constant" of integration that should be an unknown function of y.

Step 2. Integrate N with respect to y, omitting the "constant" of integration that should be an unknown function of x.

Step 3. Add the results from steps 1 and 2 together, except that if the same term (say $3xy$) appears in the answers to both steps 1 and 2, only count it once instead of twice. The result of this "addition" is what you plan to call $F(x,y)$.

The trouble is that unlike the method discussed in class and in the book, which *always* gives a solution (we proved it!), the method above *sometimes* gives a solution, and *sometimes* doesn't. (If you think that it always gives the right answer, try to prove it. The key problem you'll come across is deciding what you mean by a "term" in steps 1 and 2.) Would you drive a car that sometimes sped up when you hit the brakes?

Here is an example in which the method above fails: Solve the equation

$$(5y\sin^3 x\cos x+y-3y\sin x\cos^3 x-y\sin x\cos x)dx+(3y^2+x-2\sin^2 x\cos^2 x)dy=0. \quad (1)$$

We have $M_y=5\sin^3 x\cos x+1-3\sin x\cos^3 x-\sin x\cos x$, while $N_x=1-4\sin x\cos^3 x+4\sin^3 x\cos x$. While these *look* different, we have

$$\begin{aligned} M_y-N_x &= \sin^3 x\cos x+\sin x\cos^3 x-\sin x\cos x \\ &= \sin x\cos x(\sin^2 x+\cos^2 x-1) \\ &= 0. \end{aligned}$$

Hence $M_y=N_x$, and equation (1) is exact on the whole xy plane. Let's try to solve it the *wrong* way given above.

Step 1. We compute $\int M(x,y)dx=\frac{5}{4}y\sin^4 x+xy+\frac{3}{4}y\cos^4 x-\frac{1}{2}y\sin^2 x$.

Step 2. We compute $\int N(x,y)dy=y^3+xy-2y\sin^2 x\cos^2 x$.

Step 3. The answers to Steps 1 and 2 have only the term xy in common, so when we combine them we obtain $F(x,y)=\frac{5}{4}y\sin^4 x+xy+\frac{3}{4}y\cos^4 x-\frac{1}{2}y\sin^2 x+y^3-2y\sin^2 x\cos^2 x$. But with this F, we have

$$\begin{aligned} \frac{\partial F}{\partial x} &= 5y\sin^3 x\cos x+y-3y\cos^3 x\sin x-y\sin x\cos x \\ &\quad -2y[2\sin x\cos x\cos^2 x+(\sin^2 x)2\cos x(-\sin x)] \\ &= M(x,y)-4y\sin x\cos^3 x+4y\sin^3 x\cos x \\ &\neq M(x,y) \end{aligned}$$

since $4y(-\sin x\cos^3 x + \sin^3 x\cos x) = 4y\sin x\cos x(-\cos^2 x + \sin^2 x)$ is not identically zero.

Now let's find an F correctly:

Step 1. We compute

$$F(x,y) = \int M(x,y)dx = \frac{5}{4}y\sin^4 x + xy + \frac{3}{4}y\cos^4 x - \frac{1}{2}y\sin^2 x + g(y), \tag{2}$$

where g is an unknown function to be determined.

Step 2. With the F from Step 1, we compute $\frac{\partial F}{\partial y}$, set it equal to $N(x,y)$, and find $g'(y)$ (knowing in advance that since $Mdx+Ndy$ passed the exactness test, we will be able to express $g'(y)$ *in terms of* y *alone*—no x's—if we're persistent and make no mistakes):

$$\frac{\partial F}{\partial y} = \frac{5}{4}\sin^4 x + x + \frac{3}{4}\cos^4 x - \frac{1}{2}\sin^2 x + g'(y) = 3y^2 + x - 2\sin^2 x\cos^2 x,$$

$$\begin{aligned}
\iff g'(y) &= 3y^2 - 2\sin^2 x\cos^2 x - \frac{5}{4}\sin^4 x - \frac{3}{4}\cos^4 x + \frac{1}{2}\sin^2 x \\
&= 3y^2 - 2\sin^2 x(1-\sin^2 x) - \frac{5}{4}\sin^4 x - \frac{3}{4}(1-\sin^2 x)^2 + \frac{1}{2}\sin^2 x \\
&= 3y^2 - 2\sin^2 x(1-\sin^2 x) - \frac{5}{4}\sin^4 x - \frac{3}{4}(1-\sin^2 x)^2 + \frac{1}{2}\sin^2 x \\
&= 3y^2 - 2\sin^2 x + 2\sin^4 x - \frac{5}{4}\sin^4 x - \frac{3}{4}(1 - 2\sin^2 x + \sin^4 x) + \frac{1}{2}\sin^2 x \\
&= 3y^2 + \sin^2 x(-2 + \frac{3}{2} + \frac{1}{2}) + \sin^4 x(2 - \frac{5}{4} - \frac{3}{4}) - \frac{3}{4} \\
&= 3y^2 - \frac{3}{4}\ .
\end{aligned}$$

Step 3. Integrate to get a specific antiderivative of $g'(y)$:

$$g(y) = \int_{\text{spec}} g'(y)dy = \int_{\text{spec}} (3y^2 - \frac{3}{4})dy = y^3 - \frac{3}{4}\,y.$$

Step 4. Substitute this $g(y)$ into (2) to get an F for which $dF = Mdx + Ndy$:

$$F(x,y) = \frac{5}{4}y\sin^4 x + xy + \frac{3}{4}y\cos^4 x - \frac{1}{2}y\sin^2 x + y^3 - \frac{3}{4}\,y. \tag{3}$$

This F is *guaranteed* to have $\frac{\partial F}{\partial x} = M(x,y)$ and $\frac{\partial F}{\partial y} = N(x,y)$, provided we made no mistakes in our algebra or integration. Of course, it is always worthwhile to check. This check is left as an exercise (you will need to use the identity $\sin^2 x + \cos^2 x = 1$ again).

We are done finding F, but we have not yet given an answer to the original "Solve the equation" problem. That answer (written in the form coming most quickly from the work above; there are other equivalent forms) is

$$\left\{\frac{5}{4}y\sin^4 x + xy + \frac{3}{4}y\cos^4 x - \frac{1}{2}y\sin^2 x + y^3 - \frac{3}{4}\,y = C\right\}. \tag{4}$$