Non-book problems

1. Verify that for any nonzero constant b, the function $f(x) = \frac{1}{b}\cosh(bx)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - b\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0.$$

(Recall that the function "cosh" is defined by $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

2. Consider the equation

$$x^2 + y^2 = 4. \tag{1}$$

On the interval $-2 < x < 2$, there are two continuous functions of x determined by (1). These may be expressed by the equations $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. Show directly, without implicit differentiation, that each of these two equations is an explicit solution of the differential equation $x + y\ dy/dx = 0$. (Note that this is the DE obtained by implicitly differentiating (1) with respect to x, and then dividing by 2 just to simplify.)

3. Solve the equation

$$\frac{dy}{dx} = \frac{x \sin x}{\ln y} \quad .$$

4. Solve the equation

$$\frac{dy}{dx} = \frac{\tan^{-1} x}{ye^{2y}} \quad .$$

(*Notational reminder*: "$\tan^{-1}$" denotes the inverse-tangent function, also called arctangent, and also written "arctan". It does *not* denote the reciprocal of the tangent function, which is the cotangent function "cot".)

5. Let p be a function that is differentiable on the whole real line, and consider the separable differential equation

$$\frac{dy}{dx} = p(y). \tag{2}$$

(Here, the function $g(x)$ that you're used to seeing is just the constant function 1.)

(a) Show that the family of all solutions of (2) is *translation-invariant* in the following sense: if $y = \phi(x)$ is a solution on an interval $a < x < b$, and k is any constant, then $y = \phi(x - k)$ is a solution on the interval $a + k < x < b + k$. (Said another way: horizontally translating the graph of a solution by any amount, you get the graph of another solution.)

(b) Using the Fundamental Existence/Uniqueness Theorem for First-Order Initial-Value Problems (Theorem 1 on p. 11 in the textbook), show that for every point $(x_0, y_0) \in \mathbf{R}^2$, the initial-value problem

$$\frac{dy}{dx} = p(y), \quad y(x_0) = y_0, \tag{3}$$

has a unique solution on some open interval containing x_0.

(c) Assume that there are numbers $c < d$ such that $p(c) = p(d) = 0$. Use the "Uniqueness" part of the Fundamental Existence/Uniqueness Theorem to show all of the following. (Once you see how to do any one of these, the other two should be easy.)

- (i) If $y_0 > d$, and ϕ is a solution of (3) defined on an open interval I_{x_0} containing x_0, then $\phi(x) > d$ for all $x \in I_{x_0}$. (Note: you are not allowed to assume that I_{x_0} is a *small* interval; you have to show that what's stated is true no matter how large I_{x_0} is. The interval I_{x_0} could even be the whole real line.)

- (ii) If $y_0 < c$, then the solution ϕ of (3) satisfies $\phi(x) < c$ for all $x \in I_{x_0}$. (Same note as above applies.)

- (iii) If $c < y_0 < d$, then the solution ϕ of (3) satisfies $c < \phi(x) < d$ for all $x \in I_{x_0}$. (Same note as above applies.)

6. Solve the differential equation $\frac{dy}{dx} = 2xy(1 - y^2)$.

7. For the differential equation $\frac{dx}{dt} = x^2 - 4$ (whose general solution was found in class), solve the initial-value problem with each of the following initial conditions: (a) $x(0) = 2$; (b) $x(0) = 1$; (c) $x(0) = -2$; (d) $x(0) = -3$; (e) $x(-\frac{1}{2}\ln 5) = 3$. In each case, state the domain of the maximal solution.

8. Solve the following differential equations.
   (a) $\frac{du}{dt} + \frac{2}{t}u = e^t, \quad t < 0$.

   (b) $\frac{dy}{dx} - (\tan x)y = \sec x \ln x, \quad 0 < x < \pi/2$.

   (c) $x^2\frac{dy}{dx} - 3xy = x^6 \tan^{-1} x$.

9. Show that if F_1 and F_2 are continuously differentiable functions on an open rectangle R in the xy plane, and $dF_2 = dF_1$ throughout R, then F_1 and F_2 differ by a constant (i.e. there is a constant C such that $F_2(x, y) = F_1(x, y) + C$ for all $(x, y) \in R$).

10. **Passing the "Exactness Test" not sufficient for exactness on domain with a hole.** As discussed in class and in the book, if M and N are continuously differentiable (i.e. have continuous first partial derivatives) on an open rectangle R in the xy plane, and $M_y = N_x$ throughout R, then $Mdx + Ndy$ is exact on R. A rectangle is an example of what mathematicians call a *simply connected* region: a region with "no holes". (The intuitive notion of "no holes" can be given a precise definition, but not in MAP 2302.) It can be shown that on

any simply connected region R, not just rectangles, if M and N are continuously differentiable, then $Mdx + Ndy$ is exact on R if and only if $M_y = N_x$ throughout R.

If R is not simply connected, then "$M_y = N_x$" is still a *necessary* condition for exactness on R, but not a *sufficient* condition: there are always differentials that satisfy $M_y = N_x$, but that are not exact. You will construct an example in this exercise. The non-simply-connected region we will use is

$$R = \{(x, y) \in \mathbf{R}^2 \mid (x, y) \neq (0, 0)\}, \tag{4}$$

i.e. $\mathbf{R}^2$ with the origin removed. This region has a "hole" at the origin. On R, define

$$M(x, y) = \frac{-y}{x^2 + y^2}\ , \quad N(x, y) = \frac{x}{x^2 + y^2}\ . \tag{5}$$

For the rest of this exercise, "R" always means the region in (4), and "M" and "N" always mean the functions in (5).

(a) Show that M and N are continuously differentiable on R and that $M_y = N_x$ throughout R.

(b) Show that on the set $\{(x, y) \in \mathbf{R}^2 \mid x \text{ and } y \text{ are both nonzero}\}$ (i.e. $\mathbf{R}^2$ with the coordinate axes removed),

$$M(x, y)dx + N(x, y)dy = d(\tan^{-1}(\frac{y}{x})) = d(-\tan^{-1}(\frac{x}{y})).$$

(c) Define four functions as follows, with the indicated domains.

$$\begin{aligned}
F_{\text{right}}(x, y) &= \tan^{-1}(\frac{y}{x}), \quad x > 0. \\
F_{\text{upper}}(x, y) &= -\tan^{-1}(\frac{x}{y}) + \frac{\pi}{2}, \quad y > 0. \\
F_{\text{left}}(x, y) &= \tan^{-1}(\frac{y}{x}) + \pi, \quad x < 0. \\
F_{\text{lower}}(x, y) &= -\tan^{-1}(\frac{x}{y}) + \frac{3\pi}{2}, \quad y < 0.
\end{aligned}$$

Show that the following four identities hold:

$$\begin{aligned}
F_{\text{upper}}(x, y) &= F_{\text{right}}(x, y) \text{ throughout open quadrant I.} \\
F_{\text{left}}(x, y) &= F_{\text{upper}}(x, y) \text{ throughout open quadrant II.} \\
F_{\text{lower}}(x, y) &= F_{\text{left}}(x, y) \text{ throughout open quadrant III.} \\
F_{\text{right}}(x, y) &= F_{\text{lower}}(x, y) + 2\pi \text{ throughout open quadrant IV.}
\end{aligned}$$

Quadrants I–IV are the usual quadrants of the xy plane, and "open quadrant" means "quadrant with the coordinate axes removed".

(d) Use the result of exercise 9 (of these non-book problems) to show the following:

- F_{upper} is the *only* continuously differentiable function defined on the entire open upper half-plane $\{(x, y) \in \mathbf{R}^2 : y > 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{right} on open quadrant I.
- F_{left} is the *only* continuously differentiable function defined on the entire open left half-plane $\{(x, y) \in \mathbf{R}^2 : x < 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{upper} on open quadrant II.
- F_{lower} is the *only* continuously differentiable function defined on the entire open lower half-plane $\{(x, y) \in \mathbf{R}^2 : y < 0\}$ whose differential is $M\,dx + N\,dy$ on this half-plane and that coincides with F_{left} on open quadrant III.

(e) Show that because the identities in part (c) hold, the following definition of a function F on the domain

$$\{(x, y) \in \mathbf{R}^2 : (x, y) \neq (a, 0) \text{ for any } a \geq 0\} \tag{6}$$

(i.e. $\mathbf{R}^2$ with the origin and positive x-axis removed) is unambiguous, even though within each open quadrant the definition gives two different formulas for F:

$$\begin{aligned}
F(x, y) &= F_{\text{right}}(x, y) \quad \text{in open quadrant I.} \\
F(x, y) &= F_{\text{upper}}(x, y) \quad \text{if } y > 0, \quad \text{i.e. for } (x, y) \text{ in the open upper half-plane.} \\
F(x, y) &= F_{\text{left}}(x, y) \quad \text{if } x < 0, \quad \text{i.e. for } (x, y) \text{ in the open left half-plane.} \\
F(x, y) &= F_{\text{lower}}(x, y) \quad \text{if } y < 0, \quad \text{i.e. for } (x, y) \text{ in the open lower half-plane.} \\
F(x, y) &= F_{\text{right}}(x, y) + 2\pi \quad \text{in open quadrant IV.}
\end{aligned}$$

This function has a simple geometric interpretation: $F(x, y)$ is the polar coordinate $\theta \in (0, 2\pi)$ of the point (x, y).

(f) Use part (d) to show that F is the *only* differentiable function defined on the domain (6) whose differential is $M\,dx + N\,dy$ on this domain and that coincides with F_{right} on open quadrant I.

(g) Show that for all $x_0 > 0$, $\lim_{y \to 0+} F(x_0, y) = 0$, while $\lim_{y \to 0-} F(x_0, y) = 2\pi$.

(h) Use part (g) to show that there is *no* continuous function defined on the whole domain R (see (4)) that coincides with F on the domain (6). Then, combine this fact with part (f) to show that there is *no* continuously differentiable function on R whose differential is $M\,dx + N\,dy$ on this domain and that coincides with F_{upper} on open quadrant I.

(i) Use exercise 9 to show that if G is any continuously differentiable function defined on open quadrant I for which $dG = M dx + N dy$, then, on open quadrant I, G differs from F_{upper} only by an additive constant.

(j) Use parts (h) and (i) to show that there is *no* differentiable function H defined on all of R for which $dH = Mdx + Ndy$. Thus, $Mdx + Ndy$ is not exact on R, despite satisfying $M_y = N_x$ at every point of R.

Fact: It is accepted practice to write "$d\theta$" for the differential $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on R, even though there is *no* differentiable function θ defined on all of R whose differential is $d\theta$!

11. As remarked in the textbook in the paragraph that starts at the bottom of p. 194, solving a Cauchy-Euler (pronounced "Co-she Oiler") equation on the domain-interval $(0, \infty)$ gives us a way to solve it on the domain-interval $(-\infty, 0)$ as well. In this problem we amplify the book's remark and consider some examples.

(a) Fix numbers a, b, c (with $a \neq 0$) and consider the second-order homogeneous Cauchy-Euler equation

$$at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0. \tag{7}$$

Using the Chain Rule and the substitution $t = -u$, show that (7) for $t < 0$ is equivalent to the equation

$$au^2\frac{d^2z}{du^2} + bu\frac{dz}{du} + cz = 0 \tag{8}$$

for $u > 0$, where $z(u) = y(t) = z(-t)$. Except for the names of the variables, equations (7) and (8) are the same. Use this to show that if $t \mapsto \phi(t)$ is a solution of $at^2y'' + bty' + cy = 0$ on the interval $\{t > 0\}$, then $t \mapsto \phi(-t)$ is a solution of the same DE on the interval $\{t < 0\}$, and vice-versa. Thus show that if $t \mapsto y_{\text{gen}}(t)$ is the general solution of (7) on the interval $\{t > 0\}$, then $t \mapsto y_{\text{gen}}(|t|)$ is the general solution of (7) on the interval $\{t < 0\}$, as well as on the interval $\{t > 0\}$.

(b) Find the general solution $t \mapsto y(t)$ of

$$6t^2y'' + ty' + y = 0 \tag{9}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (9) on the interval $\{t < 0\}$.

(c) Find the general solution $t \mapsto y(t)$ of

$$t^2y'' + 5ty' + 4y = 0 \tag{10}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (10) on the interval $\{t < 0\}$. (Remember that, in all these problems, since the DE names t as its independent variable, your answer must be expressed purely in terms of t, not wholly or partly in terms of any other variable you used along the way.)

(d) Find the general solution $t \mapsto y(t)$ of

$$t^2 y'' + 2ty' + y = 0 \tag{11}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (11) on the interval $\{t < 0\}$.