

## Non-book problems

1. Verify that for any nonzero constant  $b$ , the function  $f(x) = \frac{1}{b} \cosh(bx)$  satisfies the differential equation

$$\frac{d^2y}{dx^2} - b\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0.$$

(Recall that the function “cosh” is defined by  $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ .)

2. Consider the equation

$$x^2 + y^2 = 4. \tag{1}$$

On the interval  $-2 < x < 2$ , there are two continuous functions of  $x$  determined by (1). These may be expressed by the equations  $y = \sqrt{4 - x^2}$  and  $y = -\sqrt{4 - x^2}$ . Show directly, without implicit differentiation, that each of these two equations is an (explicit) solution of the differential equation  $x + y \, dy/dx = 0$ . (Note that this is the DE obtained by implicitly differentiating (1) with respect to  $x$ , and then dividing by 2 just to simplify.)

3. Solve the equation

$$\frac{dy}{dx} = \frac{x \sin x}{\ln y}.$$

4. Solve the equation

$$\frac{dy}{dx} = \frac{\tan^{-1} x}{ye^{2y}}.$$

(*Notational reminder:* “ $\tan^{-1}$ ” denotes the inverse-tangent function, also called arctangent, and also written “arctan”. It does *not* denote the reciprocal of the tangent function, which is the cotangent function “cot”.)

5. Let  $p$  be a function that is differentiable on the whole real line, and consider the separable differential equation

$$\frac{dy}{dx} = p(y). \tag{2}$$

(Here, the function  $g(x)$  that you’re used to seeing is just the constant function 1.)

(a) Show that the family of all solutions of (2) is *translation-invariant* in the following sense: if  $y = \phi(x)$  is a solution on an interval  $a < x < b$ , and  $k$  is any constant, then  $y = \phi(x - k)$  is a solution on the interval  $a + k < x < b + k$ . (Said another way: horizontally translating the graph of a solution by any amount, you get the graph of another solution.)

(b) Using the Fundamental Existence/Uniqueness theorem for first-order initial-value problems (Theorem 5.1 in “Some note on first-order ODEs”, a much-weakened version of which is

Theorem 1 on p. 11 in the textbook), show that for every point  $(x_0, y_0) \in \mathbf{R}^2$ , the initial-value problem

$$\frac{dy}{dx} = p(y), \quad y(x_0) = y_0, \quad (3)$$

has a unique solution on some open interval containing  $x_0$ .

(c) Assume that there are numbers  $c < d$  such that  $p(c) = p(d) = 0$ . Use the “Uniqueness” part of the Fundamental Existence/Uniqueness Theorem to show all of the following. (Once you see how to do any one of these, the other two should be easy.)

- (i) If  $y_0 > d$ , and  $\phi$  is a solution of (3) defined on an open interval  $I_{x_0}$  containing  $x_0$ , then  $\phi(x) > d$  for all  $x \in I_{x_0}$ . (Note: you are not allowed to assume that  $I_{x_0}$  is a *small* interval; you have to show that what’s stated is true no matter how large  $I_{x_0}$  is. The interval  $I_{x_0}$  could even be the whole real line.)
- (ii) If  $y_0 < c$ , then the solution  $\phi$  of (3) satisfies  $\phi(x) < c$  for all  $x \in I_{x_0}$ . (Same note as above applies.)
- (iii) If  $c < y_0 < d$ , then the solution  $\phi$  of (3) satisfies  $c < \phi(x) < d$  for all  $x \in I_{x_0}$ . (Same note as above applies.)

6. Solve the differential equation  $\frac{dy}{dx} = xy^2(1 - y^2)$ .

7. For the differential equation  $\frac{dx}{dt} = x^2 - 4$  (whose general solution was found in class, with different names for the variables), solve the initial-value problem with each of the following initial conditions: (a)  $x(0) = 2$ ; (b)  $x(0) = 1$ ; (c)  $x(0) = -2$ ; (d)  $x(0) = -3$ ; (e)  $x(-\frac{1}{2} \ln 5) = 3$ . In each case, state the domain of the (maximal) solution.

8. Solve the equation

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

with the initial condition  $y(0) = -1$ . What is the domain of the (maximal) solution?

9. Solve the following differential equations.

(a)  $\frac{du}{dt} + \frac{2}{t}u = e^t, \quad t < 0.$

(b)  $\frac{dy}{dx} - (\tan x)y = \sec x \ln x, \quad 0 < x < \pi/2.$

(c)  $x^2 \frac{dy}{dx} - 3xy = x^6 \tan^{-1} x.$

10. Show that if  $F_1$  and  $F_2$  are continuously differentiable functions on an open rectangle  $R$  in the  $xy$  plane, and  $dF_2 = dF_1$  throughout  $R$ , then  $F_1$  and  $F_2$  differ by a constant (i.e. there is a constant  $C$  such that  $F_2(x, y) = F_1(x, y) + C$  for all  $(x, y) \in R$ ).

11. Solve the following differential equations and/or initial-value problems.

(a)  $(s + 1)t ds - (s^2 + 1)(t^2 + 1)dt = 0$ .

(b)  $(u + v^2 - 4)\frac{dv}{du} = 2u + 1 - v$ .

(c)  $(7\theta + 6r)d\theta + 12dr = 0$ .

(d)  $(2w - xe^x)dx + x dw = 0$ .

(e)  $6x - 1 + y + (x + 3y^2)\frac{dy}{dx} = 0, \quad y(1) = 2$ .

(f)  $x dx + \frac{4y^3}{x^2+1}dy = 0$ .

12. **Passing the “Exactness Test” not sufficient for exactness on domain with a hole.** As discussed in class and in the book, if  $M$  and  $N$  are continuously differentiable (i.e. have continuous first partial derivatives) on an open rectangle  $R$  in the  $xy$  plane, and  $M_y = N_x$  throughout  $R$ , then  $Mdx + Ndy$  is exact on  $R$ . A rectangle is an example of what mathematicians call a *simply connected* region: a region with “no holes”. (The intuitive notion of “no holes” can be given a precise definition, but not in MAP 2302.) It can be shown that on any simply connected region  $R$ , not just rectangles, if  $M$  and  $N$  are continuously differentiable, then  $Mdx + Ndy$  is exact on  $R$  if and only if  $M_y = N_x$  throughout  $R$ .

If  $R$  is not simply connected, then “ $M_y = N_x$ ” is still a *necessary* condition for exactness on  $R$ , but not a *sufficient* condition: there are always differentials that satisfy  $M_y = N_x$ , but that are not exact. You will construct an example in this exercise. The non-simply-connected region we will use is

$$R = \{(x, y) \in \mathbf{R}^2 \mid (x, y) \neq (0, 0)\}, \quad (4)$$

i.e.  $\mathbf{R}^2$  with the origin removed. This region has a “hole” at the origin. On  $R$ , define

$$M(x, y) = \frac{-y}{x^2 + y^2}, \quad N(x, y) = \frac{x}{x^2 + y^2}. \quad (5)$$

For the rest of this exercise, “ $R$ ” always means the region in (4), and “ $M$ ” and “ $N$ ” always mean the functions in (5).

(a) Show that  $M$  and  $N$  are continuously differentiable on  $R$  and that  $M_y = N_x$  throughout  $R$ .

(b) Show that on the set  $\{(x, y) \in \mathbf{R}^2 \mid x \text{ and } y \text{ are both nonzero}\}$  (i.e.  $\mathbf{R}^2$  with the coordinate axes removed),

$$M(x, y)dx + N(x, y)dy = d(\tan^{-1}(\frac{y}{x})) = d(-\tan^{-1}(\frac{x}{y})).$$

(c) Define four functions as follows, with the indicated domains.

$$\begin{aligned}
F_{\text{right}}(x, y) &= \tan^{-1}\left(\frac{y}{x}\right), \quad x > 0. \\
F_{\text{upper}}(x, y) &= -\tan^{-1}\left(\frac{x}{y}\right) + \frac{\pi}{2}, \quad y > 0. \\
F_{\text{left}}(x, y) &= \tan^{-1}\left(\frac{y}{x}\right) + \pi, \quad x < 0. \\
F_{\text{lower}}(x, y) &= -\tan^{-1}\left(\frac{x}{y}\right) + \frac{3\pi}{2}, \quad y < 0.
\end{aligned}$$

Show that the following four identities hold:

$$\begin{aligned}
F_{\text{upper}}(x, y) &= F_{\text{right}}(x, y) \quad \text{throughout open quadrant I.} \\
F_{\text{left}}(x, y) &= F_{\text{upper}}(x, y) \quad \text{throughout open quadrant II.} \\
F_{\text{lower}}(x, y) &= F_{\text{left}}(x, y) \quad \text{throughout open quadrant III.} \\
F_{\text{right}}(x, y) &= F_{\text{lower}}(x, y) + 2\pi \quad \text{throughout open quadrant IV.}
\end{aligned}$$

Quadrants I–IV are the usual quadrants of the  $xy$  plane, and “open quadrant” means “quadrant with the coordinate axes removed”.

(d) Use the result of exercise 9 (of these non-book problems) to show the following:

- $F_{\text{upper}}$  is the *only* continuously differentiable function defined on the entire open upper half-plane  $\{(x, y) \in \mathbf{R}^2 : y > 0\}$  whose differential is  $M dx + N dy$  on this half-plane and that coincides with  $F_{\text{right}}$  on open quadrant I.
- $F_{\text{left}}$  is the *only* continuously differentiable function defined on the entire open left half-plane  $\{(x, y) \in \mathbf{R}^2 : x < 0\}$  whose differential is  $M dx + N dy$  on this half-plane and that coincides with  $F_{\text{upper}}$  on open quadrant II.
- $F_{\text{lower}}$  is the *only* continuously differentiable function defined on the entire open lower half-plane  $\{(x, y) \in \mathbf{R}^2 : y < 0\}$  whose differential is  $M dx + N dy$  on this half-plane and that coincides with  $F_{\text{left}}$  on open quadrant III.

(e) Show that because the identities in part (c) hold, the following definition of a function  $F$  on the domain

$$\{(x, y) \in \mathbf{R}^2 : (x, y) \neq (a, 0) \text{ for any } a \geq 0\} \tag{6}$$

(i.e.  $\mathbf{R}^2$  with the origin and positive  $x$ -axis removed) is unambiguous, even though within each open quadrant the definition gives two different formulas for  $F$ :

$$\begin{aligned}
F(x, y) &= F_{\text{right}}(x, y) \quad \text{in open quadrant I.} \\
F(x, y) &= F_{\text{upper}}(x, y) \quad \text{if } y > 0, \text{ i.e. for } (x, y) \text{ in the open upper half-plane.} \\
F(x, y) &= F_{\text{left}}(x, y) \quad \text{if } x < 0, \text{ i.e. for } (x, y) \text{ in the open left half-plane.} \\
F(x, y) &= F_{\text{lower}}(x, y) \quad \text{if } y < 0, \text{ i.e. for } (x, y) \text{ in the open lower half-plane.} \\
F(x, y) &= F_{\text{right}}(x, y) + 2\pi \quad \text{in open quadrant IV.}
\end{aligned}$$

This function has a simple geometric interpretation:  $F(x, y)$  is the polar coordinate  $\theta \in (0, 2\pi)$  of the point  $(x, y)$ .

(f) Use part (d) to show that  $F$  is the *only* differentiable function defined on the domain (6) whose differential is  $M dx + N dy$  on this domain and that coincides with  $F_{\text{right}}$  on open quadrant I.

(g) Show that for all  $x_0 > 0$ ,  $\lim_{y \rightarrow 0^+} F(x_0, y) = 0$ , while  $\lim_{y \rightarrow 0^-} F(x_0, y) = 2\pi$ .

(h) Use part (g) to show that there is *no* continuous function defined on the whole domain  $R$  (see (4)) that coincides with  $F$  on the domain (6). Then, combine this fact with part (f) to show that there is *no* continuously differentiable function on  $R$  whose differential is  $M dx + N dy$  on this domain and that coincides with  $F_{\text{upper}}$  on open quadrant I. (i) Use exercise 9 to show that if  $G$  is any continuously differentiable function defined on open quadrant I for which  $dG = M dx + N dy$ , then, on open quadrant I,  $G$  differs from  $F_{\text{upper}}$  only by an additive constant.

(j) Use parts (h) and (i) to show that there is *no* differentiable function  $H$  defined on all of  $R$  for which  $dH = M dx + N dy$ . Thus,  $M dx + N dy$  is not exact on  $R$ , despite satisfying  $M_y = N_x$  at every point of  $R$ .

**Fact:** It is accepted practice to write “ $d\theta$ ” for the differential  $\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$  on  $R$ , even though there is *no* differentiable function  $\theta$  defined on all of  $R$  whose differential is  $d\theta$ !

13. As remarked in the textbook in the paragraph that starts at the bottom of p. 194, solving a Cauchy-Euler (pronounced “Co-she Oiler”) equation on the domain-interval  $(0, \infty)$  gives us a way to solve it on the domain-interval  $(-\infty, 0)$  as well. In this problem we amplify the book’s remark and consider some examples.

(a) Fix numbers  $a, b, c$  (with  $a \neq 0$ ) and consider the second-order homogeneous Cauchy-Euler equation

$$at^2 \frac{d^2 y}{dt^2} + bt \frac{dy}{dt} + cy = 0. \tag{7}$$

Using the Chain Rule and the substitution  $t = -u$ , show that (7) for  $t < 0$  is equivalent to the equation

$$au^2 \frac{d^2z}{du^2} + bu \frac{dz}{du} + cz = 0 \quad (8)$$

for  $u > 0$ , where  $z(u) = y(t) = z(-t)$ . Except for the names of the variables, equations (7) and (8) are the same. Use this to show that if  $t \mapsto \phi(t)$  is a solution of  $at^2y'' + bty' + cy = 0$  on the interval  $\{t > 0\}$ , then  $t \mapsto \phi(-t)$  is a solution of the same DE on the interval  $\{t < 0\}$ , and vice-versa. (See this footnote<sup>1</sup> for the meaning of “ $\mapsto$ ”.) Thus show that if  $t \mapsto y_{\text{gen}}(t)$  is the general solution of (7) on the interval  $\{t > 0\}$ , then  $t \mapsto y_{\text{gen}}(|t|)$  is the general solution of (7) on the interval  $\{t < 0\}$ , as well as on the interval  $\{t > 0\}$ .

(b) Find the general solution  $t \mapsto y(t)$  of

$$6t^2y'' + ty' + y = 0 \quad (9)$$

on the interval  $\{t > 0\}$ . Then, using part (a) above, find the general solution of (9) on the interval  $\{t < 0\}$ .

(c) Find the general solution  $t \mapsto y(t)$  of

$$t^2y'' + 5ty' + 4y = 0 \quad (10)$$

on the interval  $\{t > 0\}$ . Then, using part (a) above, find the general solution of (10) on the interval  $\{t < 0\}$ . (Remember that, in all these problems, since the DE names  $t$  as its independent variable, your answer must be expressed purely in terms of  $t$ , not wholly or partly in terms of any other variable you used along the way.)

(d) Find the general solution  $t \mapsto y(t)$  of

$$t^2y'' + 2ty' + y = 0 \quad (11)$$

on the interval  $\{t > 0\}$ . Then, using part (a) above, find the general solution of (11) on the interval  $\{t < 0\}$ .

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<sup>1</sup>The symbol “ $\mapsto$ ” is read “goes to” or (in more advanced classes) “maps to”. It is simply a way of giving a name, possibly temporarily, to the domain-variable of a function, without having to name the function. For example, “ $t \mapsto \phi(-t)$ ” is a compact way of writing “the function  $\psi$  defined by  $\psi(t) = \phi(-t)$ ”, or “the function  $g$  defined by  $g(x) = \phi(-x)$ ”.