

Non-book problems

1. Verify that for any nonzero constant b , the function $f(x) = \frac{1}{b} \cosh(bx)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - b\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0.$$

(Recall that the function “cosh” is defined by $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

2. Solve the following differential equations.

(a) $\frac{du}{dt} + \frac{2}{t}u = e^t, \quad t < 0.$

(b) $\frac{dy}{dx} - (\tan x)y = \sec x \ln x, \quad 0 < x < \pi/2.$

(c) $x^2 \frac{dy}{dx} - 3xy = x^6 \arctan x.$

3. Solve the equation

$$\frac{dy}{dx} = \frac{x \sin x}{\ln y}.$$

4. Solve the equation

$$\frac{dy}{dx} = \frac{\arctan x}{ye^{2y}}.$$

5. Solve the differential equation $\frac{dy}{dx} = xy^2(1 - y^2)$.

6. Let p be a function that is continuously differentiable (i.e., for which p' exists and is continuous) on the the whole real line, and consider the separable differential equation

$$\frac{dy}{dx} = p(y). \tag{1}$$

(Above, the “ $g(x)$ ” that you’re used to seeing on the right-hand side of a separable equation is just the constant function 1. Below, the hypotheses on the function p matter only for parts (b) and (c), not for part (a). [The originally posted version of this problem omitted the assumption that p' is continuous, which didn’t affect part (a), but does affect parts (b) and (c).])

(a) Show that the family of all solutions of (1) is *translation-invariant* in the following sense: if $y = \phi(x)$ is a solution on an interval $a < x < b$, and k is any constant, then $y = \phi(x - k)$ is

a solution on the interval $a + k < x < b + k$. (Said another way: horizontally translating the graph of a solution by any amount, you get the graph of another solution.)

(b) Using the Fundamental Theorem of Ordinary Differential Equations (in the section of the handout “Some notes on first-order ODEs” that bears this name) show that for every point $(x_0, y_0) \in \mathbf{R}^2$, the initial-value problem

$$\frac{dy}{dx} = p(y), \quad y(x_0) = y_0, \quad (2)$$

has a unique solution on every sufficiently small open interval containing x_0 .

(c) Assume that there are numbers $c < d$ such that $p(c) = p(d) = 0$. Use the “Uniqueness” part of the FTODE to show all of the following. (Once you see how to do any one of these, the other two should be easy.)

- (i) If $y_0 > d$, and ϕ is a solution of (2) defined on an open interval I_{x_0} containing x_0 , then $\phi(x) > d$ for all $x \in I_{x_0}$. (Note: you are not allowed to assume that I_{x_0} is a *small* interval; you have to show that what’s stated is true no matter how large I_{x_0} is. The interval I_{x_0} could even be the whole real line.)
- (ii) If $y_0 < c$, then the solution ϕ of (2) satisfies $\phi(x) < c$ for all $x \in I_{x_0}$. (Same note as above applies.)
- (iii) If $c < y_0 < d$, then the solution ϕ of (2) satisfies $c < \phi(x) < d$ for all $x \in I_{x_0}$. (Same note as above applies.)

7. Solve the equation

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

with the initial condition $y(0) = -1$. What is the domain of the (maximal) solution?

8. For the differential equation $\frac{dx}{dt} = x^2 - 4$ (whose general solution was found in class, with different names for the variables), solve the initial-value problem with each of the following initial conditions: (a) $x(0) = 2$; (b) $x(0) = 1$; (c) $x(0) = -2$; (d) $x(0) = -3$; (e) $x(-\frac{1}{2} \ln 5) = 3$. In each case, state the domain of the (maximal) solution.

9. Show that if F_1 and F_2 are continuously differentiable functions on an open rectangle R in the xy plane, and $dF_2 = dF_1$ throughout R , then F_1 and F_2 differ by a constant (i.e. there is a constant C such that $F_2(x, y) = F_1(x, y) + C$ for all $(x, y) \in R$).

10. Solve the following differential equations and/or initial-value problems.

- (a) $(s + 1)t \, ds - (s^2 + 1)(t^2 + 1)dt = 0$.
- (b) $(u + v^2 - 4)\frac{dv}{du} = 2u + 1 - v$.
- (c) $(7\theta + 6r)d\theta + 12dr = 0$.
- (d) $(2w - xe^x)dx + x \, dw = 0$.
- (e) $6x - 1 + y + (x + 3y^2)\frac{dy}{dx} = 0, \quad y(1) = 2$.
- (f) $x \, dx + \frac{4y^3}{x^2+1}dy = 0$.

11. Passing the “Exactness Test” not sufficient for exactness on domain with a hole. As discussed in class and in the book, if M and N are continuously differentiable (i.e. have continuous first partial derivatives) on an open rectangle R in the xy plane, and $M_y = N_x$ throughout R , then $Mdx + Ndy$ is exact on R . A rectangle is an example of what mathematicians call a *simply connected* region: a region with “no holes”. (The intuitive notion of “no holes” can be given a precise definition, but not in MAP 2302.) It can be shown that on any simply connected region R , not just rectangles, if M and N are continuously differentiable, then $Mdx + Ndy$ is exact on R if and only if $M_y = N_x$ throughout R .

If R is not simply connected, then “ $M_y = N_x$ ” is still a *necessary* condition for exactness on R , but not a *sufficient* condition: there are always differentials that satisfy $M_y = N_x$, but that are not exact. You will construct an example in this exercise. The non-simply-connected region we will use is

$$R = \{(x, y) \in \mathbf{R}^2 : (x, y) \neq (0, 0)\}, \quad (3)$$

i.e. \mathbf{R}^2 with the origin removed. This region has a “hole” at the origin. On R , define

$$M(x, y) = \frac{-y}{x^2 + y^2}, \quad N(x, y) = \frac{x}{x^2 + y^2}. \quad (4)$$

For the rest of this exercise, “ R ” always means the region in (3), and “ M ” and “ N ” always mean the functions in (4).

(a) Show that M and N are continuously differentiable on R and that $M_y = N_x$ throughout R .

(b) Show that on the set $\{(x, y) \in \mathbf{R}^2 : x \text{ and } y \text{ are both nonzero}\}$ (i.e. \mathbf{R}^2 with the coordinate axes removed),

$$M(x, y)dx + N(x, y)dy = d(\arctan(\frac{y}{x})) = d(-\arctan(\frac{x}{y})).$$

(c) Define four functions as follows, with the indicated domains.

$$\begin{aligned} F_{\text{right}}(x, y) &= \arctan\left(\frac{y}{x}\right), \quad x > 0. \\ F_{\text{upper}}(x, y) &= -\arctan\left(\frac{x}{y}\right) + \frac{\pi}{2}, \quad y > 0. \\ F_{\text{left}}(x, y) &= \arctan\left(\frac{y}{x}\right) + \pi, \quad x < 0. \\ F_{\text{lower}}(x, y) &= -\arctan\left(\frac{x}{y}\right) + \frac{3\pi}{2}, \quad y < 0. \end{aligned}$$

Show that the following four identities hold:

$$\begin{aligned} F_{\text{upper}}(x, y) &= F_{\text{right}}(x, y) \quad \text{throughout open quadrant I.} \\ F_{\text{left}}(x, y) &= F_{\text{upper}}(x, y) \quad \text{throughout open quadrant II.} \\ F_{\text{lower}}(x, y) &= F_{\text{left}}(x, y) \quad \text{throughout open quadrant III.} \\ F_{\text{right}}(x, y) &= F_{\text{lower}}(x, y) + 2\pi \quad \text{throughout open quadrant IV.} \end{aligned}$$

Quadrants I–IV are the usual quadrants of the xy plane, and “open quadrant” means “quadrant with the coordinate axes removed”.

(d) Use the result of exercise 9 (of these non-book problems) to show the following:

- F_{upper} is the *only* continuously differentiable function defined on the entire open upper half-plane $\{(x, y) \in \mathbf{R}^2 : y > 0\}$ whose differential is $M dx + N dy$ on this half-plane and that coincides with F_{right} on open quadrant I.
- F_{left} is the *only* continuously differentiable function defined on the entire open left half-plane $\{(x, y) \in \mathbf{R}^2 : x < 0\}$ whose differential is $M dx + N dy$ on this half-plane and that coincides with F_{upper} on open quadrant II.
- F_{lower} is the *only* continuously differentiable function defined on the entire open lower half-plane $\{(x, y) \in \mathbf{R}^2 : y < 0\}$ whose differential is $M dx + N dy$ on this half-plane and that coincides with F_{left} on open quadrant III.

(e) Show that because the identities in part (c) hold, the following definition of a function F on the domain

$$\{(x, y) \in \mathbf{R}^2 : (x, y) \neq (a, 0) \text{ for any } a \geq 0\} \tag{5}$$

(i.e. \mathbf{R}^2 with the origin and positive x -axis removed) is unambiguous, even though within each open quadrant the definition gives two different formulas for F :

$$\begin{aligned}
F(x, y) &= F_{\text{right}}(x, y) \quad \text{in open quadrant I.} \\
F(x, y) &= F_{\text{upper}}(x, y) \quad \text{if } y > 0, \text{ i.e. for } (x, y) \text{ in the open upper half-plane.} \\
F(x, y) &= F_{\text{left}}(x, y) \quad \text{if } x < 0, \text{ i.e. for } (x, y) \text{ in the open left half-plane.} \\
F(x, y) &= F_{\text{lower}}(x, y) \quad \text{if } y < 0, \text{ i.e. for } (x, y) \text{ in the open lower half-plane.} \\
F(x, y) &= F_{\text{right}}(x, y) + 2\pi \quad \text{in open quadrant IV.}
\end{aligned}$$

This function has a simple geometric interpretation: $F(x, y)$ is the polar coordinate $\theta \in (0, 2\pi)$ of the point (x, y) .

(f) Use part (d) to show that F is the *only* differentiable function defined on the domain (5) whose differential is $M dx + N dy$ on this domain and that coincides with F_{right} on open quadrant I.

(g) Show that for all $x_0 > 0$, $\lim_{y \rightarrow 0^+} F(x_0, y) = 0$, while $\lim_{y \rightarrow 0^-} F(x_0, y) = 2\pi$.

(h) Use part (g) to show that there is *no* continuous function defined on the whole domain R (see (3)) that coincides with F on the domain (5). Then, combine this fact with part (f) to show that there is *no* continuously differentiable function on R whose differential is $M dx + N dy$ on this domain and that coincides with F_{upper} on open quadrant I.

(i) Use exercise 9 to show that if G is any continuously differentiable function defined on open quadrant I for which $dG = M dx + N dy$, then, on open quadrant I, G differs from F_{upper} only by an additive constant.

(j) Use parts (h) and (i) to show that there is *no* differentiable function H defined on all of R for which $dH = M dx + N dy$. Thus, $M dx + N dy$ is not exact on R , despite satisfying $M_y = N_x$ at every point of R .

Fact: It is accepted practice to write “ $d\theta$ ” for the differential $\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$ on R , even though there is *no* differentiable function θ defined on all of R whose differential is $d\theta$!

12. Show by induction on k that if an operator L is linear, then for all $k \geq 1$, all constants c_1, c_2, \dots, c_k , and all functions f_1, f_2, \dots, f_k in the set of functions f for which $L[f]$ is defined,

$$L[c_1 f_1 + c_2 f_2 + \dots + c_k f_k] = c_1 L[f_1] + c_2 L[f_2] + \dots + c_k L[f_k].$$

(Several weeks ago, I sketched the argument in class. I want you to go through it again on your own.)

13. Solving a Cauchy-Euler (“Co-she Oiler”) equation on the domain-interval $(0, \infty)$ gives us a way to solve it on the domain-interval $(-\infty, 0)$ as well. Below, you will see how. some examples.

(a) Fix numbers a, b, c (with $a \neq 0$) and consider the second-order homogeneous Cauchy-Euler equation

$$at^2 \frac{d^2 y}{dt^2} + bt \frac{dy}{dt} + cy = 0. \quad (6)$$

Using the Chain Rule and the substitution $t = -u$, show that (6) for $t < 0$ is equivalent to the equation

$$au^2 \frac{d^2 z}{du^2} + bu \frac{dz}{du} + cz = 0 \quad (7)$$

for $u > 0$, where $z(u) = y(t) = z(-t)$. Except for the names of the variables, equations (6) and (7) are the same. Use this to show that if $t \mapsto \phi(t)$ is a solution of $at^2 y'' + bt y' + cy = 0$ on the interval $\{t > 0\}$, then $t \mapsto \phi(-t)$ is a solution of the same DE on the interval $\{t < 0\}$, and vice-versa. (See this footnote¹ for the meaning of “ \mapsto ”.) Thus show that if $t \mapsto y_{\text{gen}}(t)$ is the general solution of (6) on the interval $\{t > 0\}$, then $t \mapsto y_{\text{gen}}(|t|)$ is the general solution of (6) on the interval $\{t < 0\}$, as well as on the interval $\{t > 0\}$.

(b) Find the general solution $t \mapsto y(t)$ of

$$6t^2 y'' + ty' + y = 0 \quad (8)$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (8) on the interval $\{t < 0\}$.

(c) Find the general solution $t \mapsto y(t)$ of

$$t^2 y'' + 5ty' + 4y = 0 \quad (9)$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (9) on the interval $\{t < 0\}$. (Remember that, in all these problems, since the DE names t as its independent variable, your answer must be expressed purely in terms of t , not wholly or partly in terms of any other variable you used along the way.)

(d) Find the general solution $t \mapsto y(t)$ of

$$t^2 y'' + 2ty' + y = 0 \quad (10)$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (10) on the interval $\{t < 0\}$.

¹Recall that symbol “ \mapsto ” is read “goes to” or (in more advanced classes) “maps to”. It is simply a way of giving a name, possibly temporarily, to the domain-variable of a function, without having to name the function. For example, “ $t \mapsto \phi(-t)$ ” is a compact way of writing “the function ψ defined by $\psi(t) = \phi(-t)$ ”.

14. Let f_1, f_2, \dots, f_n be functions on an interval I . Recall that the definition of “the set $\{f_1, f_2, \dots, f_n\}$ is linearly *dependent*” is: there exist constants c_1, c_2, \dots, c_n , not all 0 (i.e. at least one of which is nonzero), such that the function $c_1f_1 + c_2f_2 + \dots + c_nf_n$ is identically zero.

Show that $\{f_1, f_2, \dots, f_n\}$ is linearly dependent if and only if at least one of the functions f_i is a linear combination of the others.

(Note that in the case $n = 2$, the latter condition reduces to: either $f_1 = cf_2$ for some constant c , or $f_2 = cf_1$ for some constant c . This exercise is giving you the generalization of that $n = 2$ condition for linear dependence.)