# Non-book problems

1. Verify that for any nonzero constant $b$, the function $f(x) = \frac{1}{b}\cosh(bx)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - b\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0.$$

(Recall that the function "cosh" is defined by $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

2. Solve the following differential equations.

(a) $\frac{du}{dt} + \frac{2}{t}u = e^t, \quad t < 0.$

(b) $\frac{dy}{dx} - (\tan x)y = \sec x \;\ln x, \quad 0 < x < \pi/2.$

(c) $x^2\frac{dy}{dx} - 3xy = x^6 \arctan x.$

3. Solve the equation

$$\frac{dy}{dx} = \frac{x \sin x}{\ln y} \quad .$$

4. Solve the equation

$$\frac{dy}{dx} = \frac{\arctan x}{ye^{2y}} \quad .$$

5. Solve the equation $\frac{dy}{dx} = xy^2(1 - y^2)$.

6. Let $p$ be a function that is continuously differentiable (i.e., for which $p'$ exists and is continuous) on the the whole real line, and consider the separable differential equation

$$\frac{dy}{dx} = p(y). \tag{1}$$

(Above, the "$g(x)$" that you're used to seeing on the right-hand side of a separable equation is just the constant function 1. Below, the hypotheses on the function $p$ matter only for parts (b) and (c), not for part (a).

(a) Show that the family of all solutions of (1) is *translation-invariant* in the following sense: if $y = \phi(x)$ is a solution on an interval $a < x < b$, and $k$ is any constant, then $y = \phi(x - k)$ is a solution on the interval $a + k < x < b + k$. (Said another way: horizontally translating the graph of a solution by any amount, you get the graph of another solution.)

(b) Using the Fundamental Theorem of Ordinary Differential Equations (in the section of the handout "Some notes on first-order ODEs" that bears this name) show that for every point $(x_0, y_0) \in \mathbf{R}^2$, the initial-value problem

$$\frac{dy}{dx} = p(y), \quad y(x_0) = y_0, \tag{2}$$

has a unique solution on every sufficiently small open interval containing $x_0$.

(c) Assume that there are numbers $c < d$ such that $p(c) = p(d) = 0$. Use the "Uniqueness" part of the FTODE to show all of the following. (Once you see how to do any one of these, the other two should be easy.)

- (i) If $y_0 > d$, and $\phi$ is a solution of (2) defined on an open interval $I_{x_0}$ containing $x_0$, then $\phi(x) > d$ for all $x \in I_{x_0}$. (Note: you are not allowed to assume that $I_{x_0}$ is a *small* interval; you have to show that what's being stated above is true no matter how large $I_{x_0}$ is. The interval $I_{x_0}$ could even be the whole real line.)
- (ii) If $y_0 < c$, then the solution $\phi$ of (2) satisfies $\phi(x) < c$ for all $x \in I_{x_0}$. (Same note as above applies.)
- (iii) If $c < y_0 < d$, then the solution $\phi$ of (2) satisfies $c < \phi(x) < d$ for all $x \in I_{x_0}$. (Same note as above applies.)

7. Solve the equation

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

with the initial condition $y(0) = -1$. What is the domain of the (maximal) solution?

8. For the differential equation $\frac{dx}{dt} = x^2 - 4$ (whose general solution was found in class [or will be, by the time this problem is assigned], possibly with different names for the variables), solve the initial-value problem with each of the following initial conditions: (a) $x(0) = 2$; (b) $x(0) = 1$; (c) $x(0) = -2$; (d) $x(0) = -3$; (e) $x(-\frac{1}{2}\ln 5) = 3$. In each case, state the domain of the (maximal) solution.

9. Show that if $F_1$ and $F_2$ are continuously differentiable functions on an open rectangle $R$ in the $xy$ plane, and $dF_2 = dF_1$ throughout $R$, then $F_1$ and $F_2$ differ by a constant (i.e. there is a constant $C$ such that $F_2(x, y) = F_1(x, y) + C$ for all $(x, y) \in R$).

10. Solve the following differential equations and/or initial-value problems.

(a) $(s+1)t\ ds - (s^2+1)(t^2+1)dt = 0$.

(b) $(u+v^2-4)\frac{dv}{du} = 2u+1-v$.

(c) $(7\theta+6r)d\theta + 12dr = 0$.

(d) $(2w - xe^x)dx + x\ dw = 0$.

(e) $6x - 1 + y + (x+3y^2)\frac{dy}{dx} = 0, \quad y(1) = 2$.

(f) $x\ dx + \frac{4y^3}{x^2+1}dy = 0$.

11. **Passing the "Exactness Test" not sufficient for exactness on domain with a hole.** As discussed in class and in the book, if $M$ and $N$ are continuously differentiable (i.e. have continuous first partial derivatives) on an open rectangle $R$ in the $xy$ plane, and $M_y = N_x$ throughout $R$, then $Mdx + Ndy$ is exact on $R$. A rectangle is an example of what mathematicians call a *simply connected* region: a region with "no holes". (The intuitive notion of "no holes" can be given a precise definition, but not in MAP 2302.) It can be shown that on any simply connected region $R$, not just rectangles, if $M$ and $N$ are continuously differentiable, then $Mdx + Ndy$ is exact on $R$ if and only if $M_y = N_x$ throughout $R$.

If $R$ is not simply connected, then "$M_y = N_x$" is still a *necessary* condition for exactness on $R$, but not a *sufficient* condition: there are always differentials that satisfy $M_y = N_x$, but that are not exact. You will construct an example in this exercise. The non-simply-connected region we will use is

$$R = \{(x,y) \in \mathbf{R}^2 : (x,y) \neq (0,0)\}, \tag{3}$$

i.e. $\mathbf{R}^2$ with the origin removed. This region has a "hole" at the origin. On $R$, define

$$M(x,y) = \frac{-y}{x^2+y^2}\ , \quad N(x,y) = \frac{x}{x^2+y^2}\ . \tag{4}$$

For the rest of this exercise, "$R$" always means the region in (3), and "$M$" and "$N$" always mean the functions in (4).

(a) Show that $M$ and $N$ are continuously differentiable on $R$ and that $M_y = N_x$ throughout $R$.

(b) Show that on the set $\{(x,y) \in \mathbf{R}^2 : x \text{ and } y \text{ are both nonzero}\}$ (i.e. $\mathbf{R}^2$ with the coordinate axes removed),

$$M(x,y)dx + N(x,y)dy = d(\arctan(\frac{y}{x})) = d(-\arctan(\frac{x}{y})).$$

(c) Define four functions as follows, with the indicated domains.

$$\begin{aligned} F_{\text{right}}(x,y) &= \arctan(\frac{y}{x}), \quad x>0. \\ F_{\text{upper}}(x,y) &= -\arctan(\frac{x}{y})+\frac{\pi}{2}, \quad y>0. \\ F_{\text{left}}(x,y) &= \arctan(\frac{y}{x})+\pi, \quad x<0. \\ F_{\text{lower}}(x,y) &= -\arctan(\frac{x}{y})+\frac{3\pi}{2}, \quad y<0. \end{aligned}$$

Show that the following four identities hold:

$$\begin{aligned} F_{\text{upper}}(x,y) &= F_{\text{right}}(x,y) \text{ throughout open quadrant I.} \\ F_{\text{left}}(x,y) &= F_{\text{upper}}(x,y) \text{ throughout open quadrant II.} \\ F_{\text{lower}}(x,y) &= F_{\text{left}}(x,y) \text{ throughout open quadrant III.} \\ F_{\text{right}}(x,y) &= F_{\text{lower}}(x,y)+2\pi \text{ throughout open quadrant IV.} \end{aligned}$$

Quadrants I–IV are the usual quadrants of the $xy$ plane, and "open quadrant" means "quadrant with the coordinate axes removed".

(d) Use the result of exercise 9 (of these non-book problems) to show the following:

- $F_{\text{upper}}$ is the *only* continuously differentiable function defined on the entire open upper half-plane $\{(x,y)\in \mathbf{R}^2 : y>0\}$ whose differential is $M\,dx+N\,dy$ on this half-plane and that coincides with $F_{\text{right}}$ on open quadrant I.
- $F_{\text{left}}$ is the *only* continuously differentiable function defined on the entire open left half-plane $\{(x,y)\in \mathbf{R}^2 : x<0\}$ whose differential is $M\,dx+N\,dy$ on this half-plane and that coincides with $F_{\text{upper}}$ on open quadrant II.
- $F_{\text{lower}}$ is the *only* continuously differentiable function defined on the entire open lower half-plane $\{(x,y)\in \mathbf{R}^2 : y<0\}$ whose differential is $M\,dx+N\,dy$ on this half-plane and that coincides with $F_{\text{left}}$ on open quadrant III.

(e) Show that because the identities in part (c) hold, the following definition of a function $F$ on the domain

$$\{(x,y)\in \mathbf{R}^2 : (x,y)\neq (a,0) \text{ for any } a\geq 0\} \tag{5}$$

(i.e. $\mathbf{R}^2$ with the origin and positive $x$-axis removed) is unambiguous, even though within each open quadrant the definition gives two different formulas for $F$:

$$
\begin{aligned}
F(x,y) &= F_{\text{right}}(x,y) \quad \text{in open quadrant I.} \\
F(x,y) &= F_{\text{upper}}(x,y) \quad \text{if } y>0, \quad \text{i.e. for } (x,y) \text{ in the open upper half-plane.} \\
F(x,y) &= F_{\text{left}}(x,y) \quad \text{if } x<0, \quad \text{i.e. for } (x,y) \text{ in the open left half-plane.} \\
F(x,y) &= F_{\text{lower}}(x,y) \quad \text{if } y<0, \quad \text{i.e. for } (x,y) \text{ in the open lower half-plane.} \\
F(x,y) &= F_{\text{right}}(x,y)+2\pi \quad \text{in open quadrant IV.}
\end{aligned}
$$

This function has a simple geometric interpretation: $F(x,y)$ is the polar coordinate $\theta \in (0, 2\pi)$ of the point $(x,y)$.

(f) Use part (d) to show that $F$ is the *only* differentiable function defined on the domain (5) whose differential is $M\,dx + N\,dy$ on this domain and that coincides with $F_{\text{right}}$ on open quadrant I.

(g) Show that for all $x_0 > 0$, $\lim_{y\to 0+} F(x_0, y) = 0$, while $\lim_{y\to 0-} F(x_0, y) = 2\pi$.

(h) Use part (g) to show that there is *no* continuous function defined on the whole domain $R$ (see (3)) that coincides with $F$ on the domain (5). Then, combine this fact with part (f) to show that there is *no* continuously differentiable function on $R$ whose differential is $M\,dx+N\,dy$ on this domain and that coincides with $F_{\text{upper}}$ on open quadrant I.

(i) Use exercise 9 to show that if $G$ is any continuously differentiable function defined on open quadrant I for which $dG = Mdx + Ndy$, then, on open quadrant I, $G$ differs from $F_{\text{upper}}$ only by an additive constant.

(j) Use parts (h) and (i) to show that there is *no* differentiable function $H$ defined on all of $R$ for which $dH = Mdx + Ndy$. Thus, $Mdx + Ndy$ is not exact on $R$, despite satisfying $M_y = N_x$ at every point of $R$.

**Fact**: It is accepted practice to write "$d\theta$" for the differential $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on $R$, even though there is *no* differentiable function $\theta$ defined on all of $R$ whose differential is $d\theta$!

12. Show by induction on $k$ that if an operator $L$ is linear, then for all $k \geq 1$, all constants $c_1, c_2, \ldots, c_k$, and all functions $f_1, f_2, \ldots f_k$ in the set of functions $f$ for which $L[f]$ is defined,

$$L[c_1f_1 + c_2f_2 + \cdots + c_kf_k] = c_1L[f_1] + c_2L[f_2] + \cdots + c_kL[f_k].$$

(Several weeks ago, I sketched the argument in class. I want you to go through it again on your own.)

13. Solving a Cauchy-Euler ("Co-she Oiler") equation on the domain-interval $(0,\infty)$ gives us a way to solve it on the domain-interval $(-\infty, 0)$ as well. Below, you will see how. some examples.

(a) Fix numbers $a, b, c$ (with $a \neq 0$) and consider the second-order homogeneous Cauchy-Euler equation

$$at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0. \tag{6}$$

Using the Chain Rule and the substitution $t = -u$, show that (6) for $t < 0$ is equivalent to the equation

$$au^2\frac{d^2z}{du^2} + bu\frac{dz}{du} + cz = 0 \tag{7}$$

for $u > 0$, where $z(u) = y(t) = z(-t)$. Except for the names of the variables, equations (6) and (7) are the same. Use this to show that if $t \mapsto \phi(t)$ is a solution of $at^2y'' + bty' + cy = 0$ on the interval $\{t > 0\}$, then $t \mapsto \phi(-t)$ is a solution of the same DE on the interval $\{t < 0\}$, and vice-versa. (See this footnote[1] for the meaning of "$\mapsto$".) Thus show that if $t \mapsto y_{\text{gen}}(t)$ is the general solution of (6) on the interval $\{t > 0\}$, then $t \mapsto y_{\text{gen}}(|t|)$ is the general solution of (6) on the interval $\{t < 0\}$, as well as on the interval $\{t > 0\}$.

(b) Find the general solution $t \mapsto y(t)$ of

$$6t^2y'' + ty' + y = 0 \tag{8}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (8) on the interval $\{t < 0\}$.

(c) Find the general solution $t \mapsto y(t)$ of

$$t^2y'' + 5ty' + 4y = 0 \tag{9}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (9) on the interval $\{t < 0\}$. (Remember that, in all these problems, since the DE names $t$ as its independent variable, your answer must be expressed purely in terms of $t$, not wholly or partly in terms of any other variable you used along the way.)

(d) Find the general solution $t \mapsto y(t)$ of

$$t^2y'' + 2ty' + y = 0 \tag{10}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (10) on the interval $\{t < 0\}$.

[1] Recall that symbol "$\mapsto$" is read "goes to" or (in more advanced classes) "maps to". It is simply a way of giving a name, possibly temporarily, to the domain-variable of a function, without having to name the function. For example, "$t \mapsto \phi(-t)$" is a compact way of writing "the function $\psi$ defined by $\psi(t) = \phi(-t)$".