Non-book problems

1. Verify that for any nonzero constant b, the function $f(x) = \frac{1}{b}\cosh(bx)$ satisfies the differential equation

$$\frac{d^2y}{dx^2} - b\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = 0.$$

(Recall that the function "cosh" is defined by $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$.)

2. Solve the following differential equations.

(a) $\frac{du}{dt} + \frac{2}{t}u = e^t, \quad t < 0.$

(b) $\frac{dy}{dx} - (\tan x)y = \sec x \; \ln x, \quad 0 < x < \pi/2.$

(c) $x^2 \frac{dy}{dx} - 3xy = x^6 \arctan x.$

3. Solve the equation

$$\frac{dy}{dx} = \frac{x \sin x}{\ln y} \quad .$$

4. Solve the equation

$$\frac{dy}{dx} = \frac{\arctan x}{ye^{2y}} \quad .$$

5. Solve the equation $\frac{dy}{dx} = xy^2(1 - y^2)$.

6. Let p be a function that is continuously differentiable (i.e., for which p' exists and is continuous) on the the whole real line, and consider the separable differential equation

$$\frac{dy}{dx} = p(y). \tag{1}$$

(Above, the "$g(x)$" that you're used to seeing on the right-hand side of a separable equation is just the constant function 1. Below, the hypotheses on the function p matter only for parts (b) and (c), not for part (a).

(a) Show that the family of all solutions of (1) is *translation-invariant* in the following sense: if $y = \phi(x)$ is a solution on an interval $a < x < b$, and k is any constant, then $y = \phi(x - k)$ is a solution on the interval $a + k < x < b + k$. (Said another way: horizontally translating the graph of a solution by any amount, you get the graph of another solution.)

(b) Using the Fundamental Theorem of Ordinary Differential Equations (in the section of the handout "Some notes on first-order ODEs" that bears this name) show that for every point $(x_0, y_0) \in \mathbf{R}^2$, the initial-value problem

$$\frac{dy}{dx} = p(y), \quad y(x_0) = y_0, \tag{2}$$

has a unique solution on every sufficiently small open interval containing x_0.

(c) Assume that there are numbers $c < d$ such that $p(c) = p(d) = 0$. Use the "Uniqueness" part of the FTODE to show all of the following. (Once you see how to do any one of these, the other two should be easy.)

- (i) If $y_0 > d$, and ϕ is a solution of (2) defined on an open interval I_{x_0} containing x_0, then $\phi(x) > d$ for all $x \in I_{x_0}$. (Note: you are not allowed to assume that I_{x_0} is a *small* interval; you have to show that what's being stated above is true no matter how large I_{x_0} is. The interval I_{x_0} could even be the whole real line.)
- (ii) If $y_0 < c$, then the solution ϕ of (2) satisfies $\phi(x) < c$ for all $x \in I_{x_0}$. (Same note as above applies.)
- (iii) If $c < y_0 < d$, then the solution ϕ of (2) satisfies $c < \phi(x) < d$ for all $x \in I_{x_0}$. (Same note as above applies.)

7. Solve the equation

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}$$

with the initial condition $y(0) = -1$. What is the domain of the (maximal) solution?

8. For the differential equation $\frac{dx}{dt} = x^2 - 4$ (whose general solution was found in class [or will be, by the time this problem is assigned], possibly with different names for the variables), solve the initial-value problem with each of the following initial conditions: (a) $x(0) = 2$; (b) $x(0) = 1$; (c) $x(0) = -2$; (d) $x(0) = -3$; (e) $x(-\frac{1}{2}\ln 5) = 3$. In each case, state the domain of the (maximal) solution.

9. Show that if F_1 and F_2 are continuously differentiable functions on an open rectangle R in the xy plane, and $dF_2 = dF_1$ throughout R, then F_1 and F_2 differ by a constant (i.e. there is a constant C such that $F_2(x, y) = F_1(x, y) + C$ for all $(x, y) \in R$).

10. Solve the following differential equations and/or initial-value problems.

(a) $(s+1)t\ ds - (s^2+1)(t^2+1)dt = 0$.

(b) $(u+v^2-4)\frac{dv}{du} = 2u+1-v$.

(c) $(7\theta+6r)d\theta + 12dr = 0$.

(d) $(2w - xe^x)dx + x\ dw = 0$.

(e) $6x-1+y+(x+3y^2)\frac{dy}{dx} = 0, \quad y(1) = 2$.

(f) $x\ dx + \frac{4y^3}{x^2+1}dy = 0$.

11. **Passing the "Exactness Test" not sufficient for exactness on domain with a hole.** As discussed in class and in the book, if M and N are continuously differentiable (i.e. have continuous first partial derivatives) on an open rectangle R in the xy plane, and $M_y = N_x$ throughout R, then $Mdx + Ndy$ is exact on R. A rectangle is an example of what mathematicians call a *simply connected* region: a region with "no holes". (The intuitive notion of "no holes" can be given a precise definition, but not in MAP 2302.) It can be shown that on any simply connected region R, not just rectangles, if M and N are continuously differentiable, then $Mdx + Ndy$ is exact on R if and only if $M_y = N_x$ throughout R.

If R is not simply connected, then "$M_y = N_x$" is still a *necessary* condition for exactness on R, but not a *sufficient* condition: there are always differentials that satisfy $M_y = N_x$, but that are not exact. You will construct an example in this exercise. The non-simply-connected region we will use is

$$R = \{(x,y) \in \mathbf{R}^2 : (x,y) \neq (0,0)\}, \tag{3}$$

i.e. $\mathbf{R}^2$ with the origin removed. This region has a "hole" at the origin. On R, define

$$M(x,y) = \frac{-y}{x^2+y^2}\ , \quad N(x,y) = \frac{x}{x^2+y^2}\ . \tag{4}$$

For the rest of this exercise, "R" always means the region in (3), and "M" and "N" always mean the functions in (4).

(a) Show that M and N are continuously differentiable on R and that $M_y = N_x$ throughout R.

(b) Show that on the set $\{(x,y) \in \mathbf{R}^2 : x \text{ and } y \text{ are both nonzero}\}$ (i.e. $\mathbf{R}^2$ with the coordinate axes removed),

$$M(x,y)dx + N(x,y)dy = d(\arctan(\frac{y}{x})) = d(-\arctan(\frac{x}{y})).$$

(c) Define four functions as follows, with the indicated domains.

$$\begin{aligned}
F_{\text{right}}(x,y) &= \arctan(\frac{y}{x}), \quad x>0. \\
F_{\text{upper}}(x,y) &= -\arctan(\frac{x}{y})+\frac{\pi}{2}, \quad y>0. \\
F_{\text{left}}(x,y) &= \arctan(\frac{y}{x})+\pi, \quad x<0. \\
F_{\text{lower}}(x,y) &= -\arctan(\frac{x}{y})+\frac{3\pi}{2}, \quad y<0.
\end{aligned}$$

Show that the following four identities hold:

$$\begin{aligned}
F_{\text{upper}}(x,y) &= F_{\text{right}}(x,y) \;\; \text{throughout open quadrant I.} \\
F_{\text{left}}(x,y) &= F_{\text{upper}}(x,y) \;\; \text{throughout open quadrant II.} \\
F_{\text{lower}}(x,y) &= F_{\text{left}}(x,y) \;\; \text{throughout open quadrant III.} \\
F_{\text{right}}(x,y) &= F_{\text{lower}}(x,y)-2\pi \;\; \text{throughout open quadrant IV.}
\end{aligned}$$

Quadrants I–IV are the usual quadrants of the xy plane, and "open quadrant" means "quadrant with the coordinate axes removed".

(d) Use the result of exercise 9 (of these non-book problems) to show the following:

- F_{upper} is the *only* continuously differentiable function defined on the entire open upper half-plane $\{(x,y)\in\mathbf{R}^2 : y>0\}$ whose differential is $M\,dx+N\,dy$ on this half-plane and that coincides with F_{right} on open quadrant I.
- F_{left} is the *only* continuously differentiable function defined on the entire open left half-plane $\{(x,y)\in\mathbf{R}^2 : x<0\}$ whose differential is $M\,dx+N\,dy$ on this half-plane and that coincides with F_{upper} on open quadrant II.
- F_{lower} is the *only* continuously differentiable function defined on the entire open lower half-plane $\{(x,y)\in\mathbf{R}^2 : y<0\}$ whose differential is $M\,dx+N\,dy$ on this half-plane and that coincides with F_{left} on open quadrant III.

(e) Show that because the identities in part (c) hold, the following definition of a function F on the domain

$$\{(x,y)\in\mathbf{R}^2 : (x,y)\neq(a,0) \text{ for any } a\geq 0\} \tag{5}$$

(i.e. $\mathbf{R}^2$ with the origin and positive x-axis removed) is unambiguous, even though within each open quadrant the definition gives two different formulas for F:

$$
\begin{aligned}
F(x,y) &= F_{\text{right}}(x,y) \quad \text{in open quadrant I.} \\
F(x,y) &= F_{\text{upper}}(x,y) \quad \text{if } y>0, \quad \text{i.e. for } (x,y) \text{ in the open upper half-plane.} \\
F(x,y) &= F_{\text{left}}(x,y) \quad \text{if } x<0, \quad \text{i.e. for } (x,y) \text{ in the open left half-plane.} \\
F(x,y) &= F_{\text{lower}}(x,y) \quad \text{if } y<0, \quad \text{i.e. for } (x,y) \text{ in the open lower half-plane.} \\
F(x,y) &= F_{\text{right}}(x,y)+2\pi \quad \text{in open quadrant IV.}
\end{aligned}
$$

This function has a simple geometric interpretation: $F(x,y)$ is the polar coordinate $\theta \in (0, 2\pi)$ of the point (x,y).

(f) Use part (d) to show that F is the *only* differentiable function defined on the domain (5) whose differential is $M\,dx + N\,dy$ on this domain and that coincides with F_{right} on open quadrant I.

(g) Show that for all $x_0 > 0$, $\lim_{y\to 0+} F(x_0, y) = 0$, while $\lim_{y\to 0-} F(x_0,y) = 2\pi$.

(h) Use part (g) to show that there is *no* continuous function defined on the whole domain R (see (3)) that coincides with F on the domain (5). Then, combine this fact with part (f) to show that there is *no* continuously differentiable function on R whose differential is $M\,dx+N\,dy$ on this domain and that coincides with F_{upper} on open quadrant I.

(i) Use exercise 9 to show that if G is any continuously differentiable function defined on open quadrant I for which $dG = Mdx + Ndy$, then, on open quadrant I, G differs from F_{upper} only by an additive constant.

(j) Use parts (h) and (i) to show that there is *no* differentiable function H defined on all of R for which $dH = Mdx + Ndy$. Thus, $Mdx + Ndy$ is not exact on R, despite satisfying $M_y = N_x$ at every point of R.

Fact: It is accepted practice to write "$d\theta$" for the differential $\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$ on R, even though there is *no* differentiable function θ defined on all of R whose differential is $d\theta$!

12. Show by induction on k that if an operator L is linear, then for all $k \geq 1$, all constants $c_1, c_2, \ldots, c_k$, and all functions $f_1, f_2, \ldots f_k$ in the set of functions f for which $L[f]$ is defined,

$$L[c_1 f_1 + c_2 f_2 + \cdots + c_k f_k] = c_1 L[f_1] + c_2 L[f_2] + \cdots + c_k L[f_k]. \tag{6}$$

13. Below, $I \subset \mathbf{R}$ is a fixed but arbitrary interval, and "function" means "function on I" (assumed differentiable as many times as necessary in each problem-part), "differential operator" means "differential operator on I", and "solution (of a DE)" means "solution on I (of that DE)".

Let $f_1, \ldots, f_N$ and f be complex-valued functions. We say that f is a *complex linear combination* of $f_1, \ldots, f_N$ if $f = c_1 f_1 + \cdots + c_N f_N$ for some complex numbers $c_1, \ldots, c_N$. We say that the set $\{f_1, \ldots, f_N\}$ is *linearly dependent over* $\mathbf{C}$ (the complex numbers) if there exist complex numbers $c_1, \ldots, c_N$, **not all 0**, such that $c_1 f_1 + \cdots + c_N f_N = 0$; otherwise we say $\{f_1, \ldots, f_N\}$ is *linearly independent over* $\mathbf{C}$. (Thus, similarly to the real case, $\{f_1, \ldots, f_N\}$ is linearly independent over $\mathbf{C}$ if and only if the *only* complex constants for which $c_1 f_1 + \cdots + c_N f_N = 0$ are $c_1 = c_2 = \cdots = c_N = 0$.) The same argument as for the real case shows that, if $N > 1$, "$\{f_1, \ldots, f_N\}$ is linearly dependent over $\mathbf{C}$" is equivalent to "at least one of the functions f_j is a complex linear combination of the others." However, the "symmetric" definition of linear independence (the one involving a linear combination of all N functions) is the most useful one in this problem.

In the problem-parts below, parts (a)–(c) should be done in order (earlier parts are used in later parts). Parts (d)–(g) should also be done in order, but do not rely on parts (a)–(c). Parts (h)–(j) should also be done in order, but do not rely on parts (a)–(g). Part (k) should be done last; it uses parts from each of the (a)–(c), (d)–(g), and (h)–(j) groupings.

(a) Let u and v be real-valued functions and let $f = u + iv$. Observe that u and v can be expressed in terms of f and $\bar{f}$:

$$\begin{aligned} u &= \frac{f + \bar{f}}{2} = \frac{1}{2}(f + \bar{f}), \\ v &= \frac{f - \bar{f}}{2i} = -\frac{1}{2}i(f - \bar{f}). \end{aligned}$$

Aided by these relations, show that

$$\{c_1 u + c_2 v \ : \ c_1, c_2 \in \mathbf{C}\} = \{c_1 f + c_2 \bar{f} \ : \ c_1, c_2 \in \mathbf{C}\}. \tag{7}$$

Note that this amounts to showing that (i) for all $c_1, c_2 \in \mathbf{C}$, there are $c_3, c_4 \in \mathbf{C}$ such that $c_1 u + c_2 v = c_3 f + c_4 \bar{f}$, and (ii) for all $c_3, c_4 \in \mathbf{C}$ there are $c_1, c_2 \in \mathbf{C}$ such that $c_3 f + c_4 \bar{f} = c_1 u + c_2 v$. (In equaton (7), which relates two *sets* of functions, the constants c_1 and c_2 in each set of functions were "dummy variables", having no meaning outside the set-braces; we were free to use the same notation "c_1 and "c_2" within the right-hand set as within the left-hand set. However, in an equation between *functions*, such as "$c_1 u + c_2 v = c_3 f + c_4 \bar{f}$", each symbol has to have a consistent meaning; we cannot relabel c_3, c_4 back to a new c_1, c_2.)

(b) Let u, v, and f be as in part (a). Show that the following are equivalent (i.e. each implies the other two):

(i) The pair $\{u, v\}$ is linearly independent (over $\mathbf{R}$).

(ii) The pair $\{u, v\}$ is linearly independent over $\mathbf{C}$.

(iii) The pair $\{f, \bar{f}\}$ is linearly independent over $\mathbf{C}$.

(c) Let u, v, and f be as in part (a), and assume that the pair $\{u, v\}$ is linearly independent. By part (b), this assumption is equivalent to: the pair $\{f, \bar{f}\}$ is linearly independent over $\mathbf{C}$. The linear-independence assumption will be needed only for the "only if" parts of the "if and only ifs" you're asked to show below.

Clearly the set of *real* linear combinations of u and v is contained in set of *complex* linear combinations of u and v. Hence, by part (a), $\{c_1 u + c_2 v \ : \ c_1, c_2 \in \mathbf{R}\} \subset \{c_1 f + c_2 \bar{f} \ : \ c_1, c_2 \in \mathbf{C}\}$. A natural question is: *which* subset of the larger set is the smaller set? I.e., for which complex pairs (c_1, c_2) is $c_1 f + c_2 \bar{f}$ a real linear combination of u and v? Go through the following steps to answer this question.

(i) Let $c_1, c_2 \in \mathbf{C}$. Show that $c_1 u + c_2 v$ is a real-valued function if **and only if** c_1 and c_2 are real. (Suggestion: use the fact that a complex number is real if and only if its imaginary part is 0.)

(ii) Let $c_1, c_2 \in \mathbf{C}$. Show that $c_1 f + c_2 \bar{f}$ is a real-valued function if **and only if** $c_2 = \overline{c_1}$. (Suggestion: use the fact that a complex number z is real if and only if $z = \bar{z}$.)

(iii) Combine (i), (ii) and part (a) to conclude that if $c_1 f + c_2 \bar{f}$ is a real linear combination of u and v (meaning: $c_1 f + c_2 \bar{f} = c_3 u + c_4 v$ for some $c_3, c_4 \in \mathbf{R}$) if and only if $c_2 = \overline{c_1}$.

Thus,

$$\{c_1 u + c_2 v \ : \ c_1, c_2 \in \mathbf{R}\} \ = \ \{cf + \bar{c}\bar{f} \ : \ c \in \mathbf{C}\}. \tag{8}$$

Note the "conservation of parameters" in this equation: both sets are described in terms of two real parameters; on the right-hand side these are the real and imaginary parts of c. In passing from the description on the left to the description on the right, we have "traded" two real parameters for one complex parameter.

(d) Previously, we defined an n^{th}-order (real) linear differential operator L to be an operator given by

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y, \tag{9}$$

where the coefficients a_j were real-valued functions (with a_n not identically zero) and the allowed input-functions y were real-valued as well. We showed (weeks ago) that such an operator satisfies the two criteria of (real) linearity: (i) $L[f + g] = L[f] + L[g]$ and (ii) $L[cf] = cL[f]$ for all real-valued functions f and g and all real constants c. In class, we checked recently that if $n = 2$, and the coefficient functions a_j in equation (9) are real constants, then the operator L in (9) is *complex-linear* as well: properties (i) and (ii) hold for all complex-valued functions f and g and all complex constants c. Check that, more generally, the operator L in (9) is complex-linear if the a_j are allowed to be complex-valued functions (not necessarily constant), and n is allowed to be any positive integer.

We may call such an L a *complex linear differential operator*. Note that since all real numbers are also complex numbers, all real-valued functions are also complex-valued, and all real linear differential operators are also complex linear differential operators.

(e) For real linear operators, we did non-book problem 12 in class (using only the defining properties of [real] linearity, i.e. properties (i) and (ii) recalled in part (d)). Analogously, using the defining properties of complex linearity, show that if L is a complex-linear operator, then equation (6) holds for all complex-valued functions $f_1, \ldots, f_k$ and complex constants $c_1, \ldots, c_k$.

(f) Assume that L is a **real** linear differential operator. Show that if f is a complex-valued function, then

$$L[\bar{f}] = \overline{L[f]};$$

equivalently,

$$\mathrm{Re}(L[f]) = L[\mathrm{Re}(f)] \text{ and } \mathrm{Im}(L[f]) = L[\mathrm{Im}(f)], \tag{10}$$

where $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ denote the real and imaginary parts, respectively, of a complex number z. (In class we showed these facts only under the assumptions that L was second-order and had constant coefficients. These simplifying assumptions were made just so that we'd have a very concrete, limited class of differential operators in mind on a first run-through of certain material. However, you should find that the general-case argument is no harder than the special-case argument given in class.)

(g) Show that if L is a **real** linear differential operator, and u and v are real-valued functions, and $f = u + iv$, then the following are equivalent (i.e. each implies the other two):

(i) $L[f] = 0$.

(ii) $L[u] = 0 = L[(v)$.

(iii) $L[\bar{f}] = 0$.

(In class we showed this equivalence only under same simplifying assumptions as in part (f).)

(h) Consider a second-order, linear, homogeneous initial-value problem

$$(L[y] \;=)\; a_2y'' + a_1y' + a_0 = 0, \quad y(t_0) = Y_0, \; y'(t_0) = Y_1, \tag{11}$$

where the coefficient functions a_j are **real** and continuous, where a_2 is nowhere zero, and where $t_0 \in I$. Recall that the "Fundamental Theorem of *Linear* ODEs" (a theorem whose truth we are simply assuming in this course) assures us that for all *real* numbers Y_0, Y_1, equation (11) has a unique solution on every subinterval of I containing t_0 (and hence that the unique solution on any proper subinterval $J \subsetneq I$ is the retriction to J of the solution on I). Show that the same conclusion holds if Y_0 and Y_1 are arbitrary *complex* numbers, and we consider L as an operator on *complex*-valued functions.

Suggestion: Consider two real IVPs, one in which the initial conditions are $y(t_0) = \mathrm{Re}(Y_0)$, $y'(t_0) = \mathrm{Re}(Y_1)$, and one in which the initial conditions are $y(t_0) = \mathrm{Im}(Y_0)$, $y'(t_0) = \mathrm{Im}(Y_1)$. Then apply part (f) (with its conclusion written in the form (10)).

Note that part (h) implies that the set of all complex-valued solutions of $L[y] = 0$ is the same as the set of all solutions of all IVPs for $L[y] = 0$ with complex ICs based at any given t_0

(hence that the general complex-valued solution is the same as the set of all maximal-in-I solutions of all IVPs for $L[y] = 0$ with complex initial conditions based at t_0).

(i) Let L be as in part (h). Show that there is no complex solution y_1 of $L[y] = 0$ for which the general complex solution is $\{cy_1 \ : \ c \in \mathbf{C}\}$. (Suggestion: use part (h).)

(j) Let L and t_0 be as in part (h). Recall that we defined a *fundamental set of solutions* (FSS) of $L[y] = 0$ to be a list of solutions $y_1, \ldots, y_N$ such that every (real) solution of $L[y] = 0$ is a (real) linear combination of $y_1, \ldots, y_N$ (assuming that *some* such list exists, an assumption that we verified later using the solutions y_1, y_2 defined below), and such that N is as small as possible among all such lists of solutions. We define a *fundamental set of complex solutions* of $L[y] = 0$ (FSCS) analogously, simply by replacing "real" with "complex".

In class we showed that the solutions y_1, y_2 of, respectively, the IVP

$$L[y] = 0, \ y(t_0) = 1, \quad y'(t_0) = 0$$

and the IVP

$$L[y] = 0, \ y(t_0) = 0, \quad y'(t_0) = 1$$

form a fundamental set of solutions for the (real) homogeneous DE $L[y] = 0$. Show that the same set of functions $\{y_1, y_2\}$ is also a fundamental set of *complex* solutions of $L[y] = 0$. (Suggestion: to show that every complex solution can be written as a complex linear combination of y_1 and y_2, use an argument similar to the one used in class for the real case. To show that there's no set of *fewer* than two complex solutions that generates the whole set of complex solutions, use part (i).)

(k) Let L be as in part (h), and let y_1 and y_2 be real-valued functions and let $f = y_1 + iy_2$. In view of part (j), any FSCS of $L[y] = 0$ has exactly two functions. Using this fact and earlier parts of this problem, show that the following are equivalent:

(i) $\{y_1, y_2\}$ is a (real) FSS of $L[y] = 0$.

(ii) $\{f, \bar{f}\}$ is a FSCS of $L[y] = 0$.

(iii) y_1 and y_2 are solutions of $L[y] = 0$ and $\{y_1, y_2\}$ is linearly independent over $\mathbf{R}$.

(iv) f is a solution of $L[f] = 0$ and $\{f, \bar{f}\}$ is linearly independent over $\mathbf{C}$.

(There are other equivalent statements that could be added to this list. I've confined the list to the most useful equivalences.)

14. **Setup**. Let $I \subseteq \mathbf{R}$. The notation $C^\infty(I)$ denotes the space of infinitely differentiable (C^∞), real-valued functions on I.[1]

[1](1) Recall that "infinitely differentiable" means that derivatives of all orders exist. For a C^∞ function, the

We are using infinitely differentiable functions just to simplify various statements below. In particular, for a linear differential operator L on I, of any order, any $f \in C^\infty(I)$ is automatically in the domain of L (i.e. the function $L[f]$ is defined). Furthemore, if the coefficient functions in L are constant, or more generally C^∞, then the output $L[f]$ is C^∞ as well.

Below, "operator" means a (not necessarily linear) differential operator L on I with the property that whenever f is infinitely differentiable, so is $L[f]$. (What we noted at the end of the last paragraph is that any linear differential operator with C^∞ coefficients has this property.) All "input" functions f are assumed to lie in $C^\infty(I)$ (except that for an exponential function $t \mapsto e^{rt}$, this is not an *assumption*, it's a *fact*). The notation D operator ($D[f] = f'$), and for any integer $n > 1$, the notation D^n is used for the n^{th}-derivative operator. Thus, a general n^{th}-order linear differential operator L may be expressed as

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0 \ , \tag{12}$$

where the coefficient functions a_j are C^∞ (possibly constant) and the function a_n is not identically 0 on I.[2]

Part (a). For $r \in \mathbf{R}$, let $\exp_r$ be the function $t \mapsto e^{rt}$. Check that if the L in (12) has **constant coefficients**, then for each $r \in \mathbf{R}$,

$$L[\exp_r] = p_{_L}(r) \exp_r \tag{13}$$

where $p_{_L}(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0$, the *characteristic polynomial of* L in the variable r. (Equation 13 is equivalent to: $(L[\exp_r])(t) = p_{_L}(r) \exp_r(t) = p_{_L}(r) e^{rt}$. In the imprecise, "lazy" notation I use in class and that's used in the book, this equation is what I've been writing as $L[e^{rt}] = p_{_L}(r) e^{rt}$. You may continue to use the "lazy" version, and I'll continue using it in class; I just like to be more precise in my written documents.)

Thus any constant-coefficient linear differential operator can be reconstructed from its characteristic polyomial: wherever we see r in the characteristic polynomial, we just need to cross it out and replace it with D. Symbolically, we may express this fact by the equation

$$L = p_{_L}(D). \tag{14}$$

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n^{th} derviative is differentiable, hence is continuous; thus derivatives of all orders don't merely *exist*, they're continuous (the reason for the "C" in "C^∞"). (2) You may substitute the word "set" for "space". Usually we call $C^\infty(I)$ a *space* of functions, rather than just a *set* of functions, because it has the following properties:

(i) Closure under addition: for any $f, g \in C^\infty(I)$, the function $f + g$ is also in $C^\infty(I)$.

(ii) Closure under multiplication by constants (also called *scalars*): for any $f \in C^\infty(I)$ and any constant c, the function cf is also in $C^\infty(I)$.

A set of real-valued functions with these two properties is automatically a *vector space*, which I'm not defining here; I'm simply telling you the source of the word "space" in "space of infinitely differentiable, real-valued functions on I."

[2]Optionally, we can define the "zeroeth-derivative operator "D^0" by $D^0[f] = f$,and define $D^1 = D$. Then the right-hand side of (12) can be written instead as $a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D^1 + a_0 D^0$, This can serve help as a reminder that in $L[f]$, the function a_0 *multiplies* f.

Note: The only differential operators L that have characteristic polynomials are constant-coefficient linear differential operators. Characteristic polynomials are functions of *one* variable, the "r" in $\exp_r$ or "$t \mapsto e^{rt}$". The independent variable for functions on which the operator L is acting—the "t" in "$t \mapsto f(t)$"—*does not appear* in the characteristic polynomial. This is *crucial* to the close relation between the algebra of polynomials and the "algebra" of constant-coefficient linear differential operators, including relation (14).

Setup, continued. For operators L_1, L_2, define the operator L_1L_2 by

$$(L_1L_2)[f] = L_1[L_2[f]] \quad \text{for each } f. \tag{15}$$

(Thus the "product" of operators is actually a *composition*: the output of one operator, in this case L_2, is used as the input of the next, in this case L_1.) Note that the parentheses around L_1L_2 in (15) are used here just to group objects together, so that it is clear what operator is operating on f.[3] We will be using parentheses similarly for other, larger groupings later.

Part (b). Show that if operators L_1 and L_2 are linear, then so is L_1L_2.

(Do this using only the defining properties of *linearity*: $L[f+g] = L[f] + Lg$ and $L[cf] = cL[f]$ for all input-functions f, g and all constants c.)

Part (c). Show that (i) any two *constant-coefficient* linear differential operators L_1, L_2 commute (i.e. $L_1L_2 = L_2L_1$, meaning $(L_1L_2)[f] = (L_2L_1)[f]$ for all f), but that (ii) non-constant linear differential operators L_1, L_2 need not commute.

Suggestion: For (i), first see why you can reduce the problem to showing that aD^n commutes with bD^m for any $n, m \geq 0$ and any constants a, b. For (ii), it suffices to come up with *one example* of operators L_1, L_2, at least one of which does not have constant coefficients, and a function f for which $(L_1L_2)[f] \neq (L_2L_1)[f]$. Consider $L_1 = D$ and $L_2 =$ "tD", i.e. the operator for which $(L_2[f])(t) = t\big(D[f](t)\big) = tf'(t)$.

Setup, continued. If L_1, L_2, L_3 are differential operators on I, then equation (15) immediately gives us definitions of $L_1(L_2L_3)$ and $(L_1L_2)L_3$. (The parentheses are used here just to group objects together. E.g. L_2L_3 is some operator L_4, and $L_1(L_2L_3)$ means L_1L_4.)

Observe that composition of operators is associative:

$$\begin{aligned} \big(L_1\big(L_2L_3\big)\big)[f] &= L_1\left[\big(L_2L_3\big)(f)\right] &= L_1\left[L_2\big[L_3[f]\big]\right] \\ \text{while also} \quad \big(\big(L_1L_2\big)L_3\big)[f] &= (L_1L_2)\left[L_3[f]\right] &= L_1\left[L_2\big[L_3[f]\big]\right]. \end{aligned}$$

So we can simply write "$L_1L_2L_3$" for the three-fold composition without fear of ambiguity. More generally, for k operators $L_1, L_2, \ldots, L_k$ we define the k-fold composition $L_1L_2\ldots L_k$ analogously.

[3] The notation "$L_1L_2[f]$" would be less clear: if $L_2[f] = g$, then "$L_1L_2[f]$" looks like it means "L_1g," which is notation we haven't defined; we've only defined "$L_1[g]$."

Consistently with this notation, we also define "powers" of an operator: $L^2 = LL$, $L^3 = LLL$, etc. Observe that this is consistent with our notation $D^n = DD \dots D$ (with n D's) for the n^{th}-derivative operator.

Part (d). Use equation (13), show that if L_1 and L_2 are constant-coefficient operators, then

$$p_{L_1L_2}(r) = p_{L_1}(r)p_{L_2}(r). \tag{16}$$

Extend your argument to show, more generally, that if $L_1, L_2, \dots, L_k$ are constant-coefficient differential operators, then

$$p_{L_1L_2\dots L_k}(r) = p_{L_1}(r)p_{L_2}(r)\dots p_{L_k}(r). \tag{17}$$

Part (e). Deduce from part (d) that if L is a constant-coefficient differential operator, and

$$p_L(r) = a(r - r_1)(r - r_2)\dots(r - r_n) \tag{18}$$

(where the roots $r_1, \dots, r_n$ need not be distinct), then the operator L itself can be "factored" correspondingly:

$$L = a(D - r_1)(D - r_2)\dots(D - r_n). \tag{19}$$

Part (f). Check that if, instead of considering only *real*-valued C^∞ functions f, we

(i) consider C^∞ *complex*-valued functions on I,

(ii) extend our definition of the functions $\exp_r$ to allow r to be complex, and

(iii) consider constant-coefficient linear differential operators with *complex* coefficients (recalling from problem 13 that these operators L are complex-linear),

then all the arguments you gave in parts (a)–(e) still work. (Don't be lazy; actually *check* these one at a time. You should be able to do this just by re-reading your work; there should be little if anything that you need to (re)write.)

Thus, if L is a linear differential operator with *complex* constant coefficients, and $p_L(r)$ can be factored over the *complex* numbers as $a(r-r_1)(r-r_2)\dots(r-r_n)$ then it is still true that $L = a(D - r_1)(D - r_2)\dots(D - r_n)$. (*Factoring over the complex numbers* means that we allow roots and factors to be complex.)

15. Solving a Cauchy-Euler ("Co-she Oiler") equation on the domain-interval $(0, \infty)$ gives us a way to solve it on the domain-interval $(-\infty, 0)$ as well. Below, you will see how.

(a) Fix numbers a, b, c (with $a \neq 0$) and consider the second-order homogeneous Cauchy-Euler equation

$$at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = 0. \tag{20}$$

Using the Chain Rule and the substitution $t = -u$, show that (20) for $t < 0$ is equivalent to the equation

$$au^2\frac{d^2z}{du^2} + bu\frac{dz}{du} + cz = 0 \tag{21}$$

for $u > 0$, where $z(u) = y(t) = z(-t)$. Except for the names of the variables, equations (20) and (21) are the same. Use this to show that if $t \mapsto \phi(t)$ is a solution of $at^2y'' + bty' + cy = 0$ on the interval $\{t > 0\}$, then $t \mapsto \phi(-t)$ is a solution of the same DE on the interval $\{t < 0\}$, and vice-versa. (See this footnote[4] for the meaning of "$\mapsto$".) Thus show that if $t \mapsto y_{\text{gen}}(t)$ is the general solution of (20) on the interval $\{t > 0\}$, then $t \mapsto y_{\text{gen}}(|t|)$ is the general solution of (20) on the interval $\{t < 0\}$, as well as on the interval $\{t > 0\}$.

Note: We can use the same "$t = -u$" substitution to relate a *non*-homogeneous Cauchy-Euler equation

$$at^2\frac{d^2y}{dt^2} + bt\frac{dy}{dt} + cy = g(t) \tag{22}$$

on the interval $(-\infty, 0)$ (assuming that the domain of g is at least that entire interval) to one on the interval $(0, \infty)$ but we have to make the same subsitution on the right-hand side as well; in place of equation (21) we obtain

$$au^2\frac{d^2z}{du^2} + bu\frac{dz}{du} + cz = h, \tag{23}$$

where $h(u) = g(-u)$. Thus, if y is a solution of equation (22) on $(0, \infty)$, then the function $t \mapsto y(-t)$ need not be a solution of (22) on $(-\infty, 0)$.

Hence the general solution of equation (22) on $(-\infty, 0)$ is (usually) not simply $t \mapsto y_{\text{gen}}(|t|)$, where y_{gen} is the general solution of (22) on $(0, \infty)$. (However, if g happens to be an *even* function on $(-\infty, \infty)$, or on $(\infty, 0) \cup (0, \infty)$—meaning, in either case, that $g(t) = g(-t)$ for every nonzero $t \in \mathbf{R}$—then $t \mapsto y_{\text{gen}}(|t|)$ *is* the general solution of (22) on $(-\infty, 0)$ as well as on $(0, \infty)$.)[5]

(b) Find the general solution $t \mapsto y(t)$ of

$$6t^2y'' + ty' + y = 0 \tag{24}$$

[4]Recall that symbol "$\mapsto$" is read "goes to" or (in more advanced classes) "maps to". It is simply a way of giving a name, possibly temporarily, to the domain-variable of a function, without having to name the function. For example, "$t \mapsto \phi(-t)$" is a compact way of writing "the function ψ defined by $\psi(t) = \phi(-t)$".

[5]More generally, the same principles and methods apply if the domain of the function g in (22) does not contain all of $(-\infty, 0)$, but only a subset $D \subset (-\infty, 0)$ that is an interval or a union of disjoint intervals. In that case, the function h is defined only on "$-D$" $= \{-t : t \in D\} \subset (0, \infty)$. A function g whose domain is not the whole real line is *even* if, whenever $g(t)$ is defined, so is $g(-t)$, and $g(-t) = g(t)$.

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (24) on the interval $\{t < 0\}$.

(c) Find the general solution $t \mapsto y(t)$ of

$$t^2 y'' + 5ty' + 4y = 0 \tag{25}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (25) on the interval $\{t < 0\}$. (Remember that, in all these problems, since the DE names t as its independent variable, your answer must be expressed purely in terms of t, not wholly or partly in terms of any other variable you used along the way.)

(d) Find the general solution $t \mapsto y(t)$ of

$$t^2 y'' + 2ty' + y = 0 \tag{26}$$

on the interval $\{t > 0\}$. Then, using part (a) above, find the general solution of (26) on the interval $\{t < 0\}$.

(e) In part (a) we saw that the *general* solution of equation (20) on $(-\infty, 0)$ could be obtained from the general solution on $(0, \infty)$ simply by replacing t by $-t$. Is the same true for IVP's? For example, if y_1 is the solution of the DE (20) with initial conditions $y(7) = 3,\ y'(7) = 4$, then is $t \mapsto y_1(-t) := \tilde{y}_1(t)$ the solution of the IVP for the DE (20) with initial conditions $y(-7) = 3,\ y(-7) = 4$? If not, of what IVP with initial conditions at -7 is $\tilde{y}_1$ the solution?