Sets and Functions

Contents

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1 Sets

A *set* is a collection of objects called *elements*. Curly braces { } are used to display the list of elements explicitly or by description. For example,

$$\{1, 2, 15\}$$

is the set whose elements are the numbers 1, 2, and 15, while

$$\{\text{real numbers greater than } 5\}$$

is the indicated set of real numbers.

Sets can have finitely many elements (as in the first example above) or infinitely many elements (as in the second example above). In general, elements of a set can be numbers, points, vectors, functions, frogs, or anything else. But in these notes, the sets of interest will be sets of numbers, sets of points in $\mathbf{R}^n$ (usually $\mathbf{R}^2$ or $\mathbf{R}^3$), or sets of functions. Sets of functions are often referred to as *collections* of functions, or, in certain cases, *families* of functions.

Sets can have finitely many elements (as in the first example above) or infinitely many elements (as in the second example above).

Notation for elements.

The element-symbol "$\in$" is used to indicate that an object belongs to a particular set; "$\notin$" is used to indicate that an object does not belong to that set.

Set selector notation (also called "set builder notation")

When one wants to define a set by some properties of its elements, *set selector* notation is often used. Either a colon or a vertical bar may be used, as in

$$A = \{\text{doodad} : \text{doodad has property } C\}$$

and

$$A = \{\text{doodad} \mid \text{doodad has property } C\}.$$

In each case, the notation defines a set A. The vertical line and the colon are pronounced "such that". (These symbols *never* mean "equals".) Within the curly braces, the symbol that appears to the left of the colon (or vertical bar) is the notation for a typical element of A.

Examples.

- $\{x : x \text{ is a real number and } 0 < x < 5\}$.
- $\{x \mid x \text{ is a real number and } 0 < x < 5\}$.

Both examples above are read "The set of x such that x is a real number and $0 < x < 5$;" which of the two set-selector symbols you use is a matter of personal preference.

It is common to use "$\mathbf{R}$" or "$\mathbb{R}$" to denote the set of real numbers. I use $\mathbf{R}$ in printed or "e-printed" material (like this handout) and use $\mathbb{R}$ at the blackboard.[1] An alternative way of describing the set in the examples above is $\{x \in \mathbf{R} : 0 < x < 5\}$.

We use the notation $\mathbf{R}^n$ (which is read "R-n", not "R to the n") for the set of ordered n-tuples of real numbers. Thus,

$$\begin{aligned} \mathbf{R}^2 &= \{(x, y) : x, y \in \mathbf{R}\}, \\ \mathbf{R}^3 &= \{(x, y, z) : x, y, z \in \mathbf{R}\}, \\ \mathbf{R}^n &= \{(x_1, x_2, \ldots, x_n) : x_1, x_2, \ldots, x_n \in \mathbf{R}\}, \end{aligned}$$

etc.

2 Real-valued functions

A *real-valued function* f is an assignment of exactly one real number to each element of a specified set A, called the *domain* of f. The mathematical notation for this is "$f : A \to \mathbf{R}$", which is read "f, from A to $\mathbf{R}$."

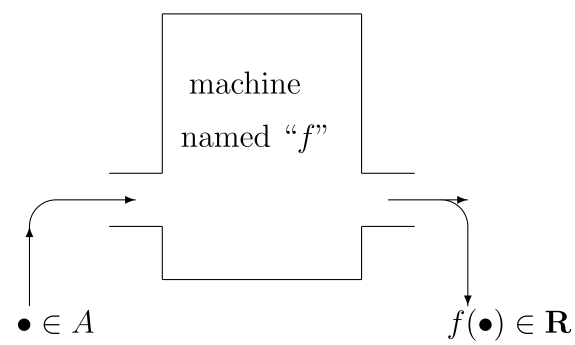
For a function $f : A \to \mathbf{R}$, the set $\mathbf{R}$ is called the *codomain* of f. The term "real-valued function" is synonymous with "function whose codomain is $\mathbf{R}$". A *real-valued function of n (real) variables* is a real-valued function whose domain is a subset of $\mathbf{R}^n$.

[1] This was standard when I was a student; the "blackboard bold" symbol $\mathbb{R}$ was invented only because true boldface couldn't be achieved with chalk.

As we move into higher mathematics, it becomes increasingly important to be able to think of a function as an *object* (so that for instance one can talk about *sets of functions*), and to understand that in Calculus-1 notation like "$f(x)$", the function is $\underline{f}$, **NOT** $f(x)$. The letter used for the *domain* variable is a *dummy variable*; it is not part of the name of the function, and can be any letter that doesn't already have a pre-assigned meaning. For example, if $f : \mathbf{R} \to \mathbf{R}$ is the function defined by $f(x) = x^2 + 2x + 3$, then $f(u) = u^2 + 2u + 3$, $f(y) = y^2 + 2y + 3$, etc.

For this purpose, a picture that is often useful is Figure 1, in which a function $f : A \to \mathbf{R}$ is viewed as a machine whose input is elements of A and whose output is elements of $\mathbf{R}$:

Figure 1: A function $f : A \to \mathbf{R}$



Being unable to divorce the domain-variable name (e.g. x) from the name of a function f causes some common mistakes when dealing with some *substitutions* (or *compositions*). For example, if again $f : \mathbf{R} \to \mathbf{R}$ is the function defined by $f(x) = x^2 + 2x + 3$, then the function $g : \mathbf{R} \to \mathbf{R}$ obtained by substituting $x - 1$ for x in "$x^2 + 2x + 3$" is given by

$$\begin{aligned} g(x) &= f(x-1) \\ &= (x-1)^2 + 2(x-1) + 3 = x^2 + 2, \end{aligned}$$

NOT by the formula "$f(x)(x - 1)$", which means "$f(x)$ **times** the quantity $x - 1$." I.e.

$$\begin{aligned} f(x)(x-1) &= (x^2 + 2x + 3)(x - 1) \\ &= x^3 + x^2 + x - 3. \end{aligned}$$

You do not have the option of changing the meaning of standard mathematical notation. The expression "$f(x)(x - 1)$" means "$f(x)$ times the quantity $x-1$" *whether or not that's what you intended.* Similarly, the function $h : \mathbf{R} \to \mathbf{R}$ obtained by substituting x^2 for x in "x^2+2x+3" is given by

$$h(x) = f(x^2) = x^4 + 2x^2 + 3,$$

NOT by the formula "$f(x)(x^2)$", which means "$f(x)$ **times** x^2." I.e.

$$f(x)(x^2) = (x^2+2x+3)x^2 = x^4 + 2x^3 + 3x^3.$$

Understanding that the function is "f", not "$f(x)$", should save you from making these common mistakes.

3 General functions

A *function* (not necessarily real-valued) is an assignment of exactly one element of a specified "target" set, called the *codomain* of the function, to each element of another set, called the *domain* of the function. Thus to specify a function, one needs three pieces of information:

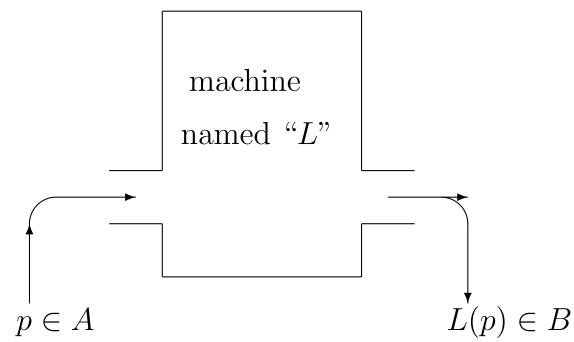
- the domain A
- the codomain B
- the "rule" specifying which element of B gets assigned to any given element of A.

A function L with domain A and codomain B is said to be a function *from A to B*. The mathematical notation for this is "$L : A \to B$", which is read "L, from A to B". For L to be a function from A to B, we require that $L(p)$ be defined for every $p \in A$, and that $L(p)$ be an element of B for every $p \in A$. We do not require that every $q \in B$ be equal to $L(p)$ for some $p \in A$. (The set of q's in B with this property is called the *range* of L.)

Note that *codomain* does not mean the same thing as *range*. In Calculus 1 and precalculus, you saw functions with different domains and ranges, but all the had the same codomain: $\mathbf{R}$. (Hence there was no need for an extra word "codomain" at that level of mathematics; since it was the same set for every function considered there.) In Calculus 3, when you spoke of vector-valued functions, "vector valued" was usually synonymous with "having codomain $\mathbf{R}^3$ or $\mathbf{R}^2$." (With so few possible codomains for the functions considered in Calculus 3, there is again no need to burden students with a new word "codomain" in that course.)

The "machine" picture of a function $L : A \to B$, generalizing Figure 1, is Figure 2, in which L is a machine whose inputs are elements of A and whose outputs elements of B:

Figure 2: A function $L : A \to B$



While this picture can be quite useful, it is not perfect: it understates (and is even misleading about) the importance of the sets A and B. By definition, *changing either of these sets changes the function.*

But an advantage of this picture is that it allows us to view some very complicated functions in more simple terms, avoiding some mental baggage. For example, let $S = \{f : \mathbf{R} \to \mathbf{R} : f \text{ is infinitely differentiable}\}$, the set of infinitely differentiable real-valued functions with domain $\mathbf{R}$. Define a function $D : S \to S$ by $D(f) = f'$ (the derivative of f). In your differential equations class, you called (or will call) D an *differential operator*. Our more abstract picture shows us that a differential operator is just another function—it simply has a domain and codomain that are different from the ones we're more used to. The domain and codomain of D are *sets of functions* $f : \mathbf{R} \to \mathbf{R}$, not subsets of $\mathbf{R}$ or $\mathbf{R}^n$ at all.

4 Difference between a function and a formula

In your calculus classes, you commonly referred to things like $\sin x, e^x, x^2 + 3x + 2$, and $\sqrt{x}$ as functions. Technically this is wrong: these are *formulas,* or *expressions*, rather than functions. They represent potential *outputs* of functions, the bottom right portions of Figures 1 and 2.

This does not mean that everything you learned in calculus is wrong! In your calculus classes, you implicitly used two principles: (i) the codomain of every function was $\mathbf{R}$, so never needed to be stated explicitly, and (ii) for the domain of the function, you used the "implied domain" of the formula: the largest set of real numbers (or the largest subset of $\mathbf{R}^2$ or $\mathbf{R}^3$) for which the formula made sense. For example, in "the function $\sqrt{x}$", it was always assumed that x was a real number, and the domain was the inteval $[0, \infty) = \{x \in \mathbf{R} \mid x \geq 0\}$. This is an example of "abuse of terminology" (a cousin of "abuse of notation"): terminology that is technically incorrect or imprecise, but (in this case) is being used because (i) the correct, precise terminology might be cumbersome and distracting (or perhaps we're just too lazy to write it!), and (ii) the terminology being used doesn't have *another* meaning that conflicts with the intended one. In place of writing "the function $\sqrt{x}$" or "the function $f(x) = \sqrt{x}$" one could more correctly (but more lengthily) write "the function $f : [0, \infty) \to \mathbf{R}$ defined by $f(x) = \sqrt{x}$."

Abusing terminology is not *always* a bad thing, *provided that both the writer and reader know what is meant* without any mind-reading, and that there is some advantage (e.g. time-saving) to using the incorrect or imprecise terminology. You will probably see most of your instructors, and the authors of your calculus and differential equations textbooks, do this when they write down functions.

One way to avoid abusing notation for commonly used functions is to give these functions a (permanent) name. For example, "sin" and "cos" are functions; *by definition* their domains and codomains are $\mathbf{R}$. (But $\sin(x)$ and $\cos(x)$ are technically not functions—they are merely the *outputs* of the "machines" "sin" and "cos" [see Figure 1] when the input is a number called x.) Another example is the exponential function $\exp : \mathbf{R} \to \mathbf{R}$, which is defined by $\exp(x) = e^x$.

But we can't reserve a permanent name for every single function. For example, there are infinitely many polynomial functions. There is a special arrow, "$\mapsto$" (pronounced "goes to" or "maps to"), that can be used when we don't want to bother introducing even a one-letter name for a function (a name that we may have no need for after defining our function). As an example of how this symbol is used: instead of writing "the function f defined by $f(x) = x^2 + 2x + 3$," we can write "the function $x \mapsto x^2 + 2x + 3$." (Here I'm assuming that we're in a context where "function" is short-hand for "function from $\mathbf{R}$ to $\mathbf{R}$.) The "$\mapsto$" symbol is often useful even when we *have* selected a name for our function.

Note that "$x \mapsto x^2 + 2x + 3$," "$t \mapsto t^2 + 2t + 3$," "$y \mapsto y^2 + 2y + 3$", etc., are all the *same function*. Again, the letter used for a function's input-variable is *not* part of the function, or even part of the function's name.