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version date: 2022-09-08

Direct Sum of Two Vector Spaces or Subspaces

There are two closely related, but not identical, notions of something called a *direct sum* in linear algebra. For one of these, we start with two vector spaces V and W, and construct a third vector space called the *direct sum* of V and W. In the other, we are given a vector space Z. and two *subspaces* V and W of Z. If these subspaces satisfy certain conditions, we say that Z is the *direct sum* of V and W. In these notes, we'll call the first notion the "external" direct sum of V and W, and denote it $V \oplus_e W$; we'll call the second notion the "internal" direct sum of V and W, and denote it $V \oplus_i W$. But conventionally, both are denoted simply $V \oplus W$; context tells you which notion is meant.

Below, we discuss both these notions, and how they are related.

"External" direct sum

Let V and W be vector spaces. On the Cartesian product $V \times W := \{(v, w) : v \in V, w \in W\}$, we define operations "plus" and "sm" (scalar[1] multiplication), and the customary notation for them, as follows:

$$\text{plus} : (V \times W) \times (V \times W) \ \rightarrow \ V \times W$$

is defined by

$$(v_1, w_1) + (v_2, w_2) := \text{plus}\big(\,(v_1, w_1), (v_2, w_2)\,\big) \ := \ (v_1 + v_2, w_1 + w_2); \quad (1)$$

scalar[2] multiplication is defined by

$$c(v, w) := \text{sm}\big(\,c, (v, w)\,\big) \ := \ (cv, cw). \quad (2)$$

(Remember: "A:=B" means that we are *defining* A to be B.)

In a homework exercise[3] you showed that $V \times W$, equipped with the operations above, is a vector space; the zero element is $(0_V, 0_W)$. This vector space is defined to be the "external" direct sum of V and W, for which we are using the notation $V \oplus_e W$ in these notes.

Remark: In equation (1) there are *three distinct operations denoted "+"*. As a general rule, it is a bad idea to use the same symbol with more than one meaning in a single equation.

[1]In these notes, "vector space" means *real* vector space; the field of scalars is $\mathbf{R}$. However, everything in these notes applies to vector spaces over *any* field $\mathbf{F}$; just replace $\mathbf{R}$ by $\mathbf{F}$ wherever $\mathbf{R}$ occurs.

[3]Friedberg, Insel, and Spence, 5th edition, exercise 1.2/ 21

However, if we try to use notation such as $+_{V\oplus_e W}$, $+_V$, and $+_W$ that distinguishes the three "+" operations from each other, the equation can be much harder to read, even if we omit the middle portion:

$$(v_1, w_1) +_{V\oplus_e W} (v_2, w_2) := (v_1 +_V v_2, w_1 +_W w_2). \tag{3}$$

Equation (2) has a similar issue—there are three distinct scalar-multiplication operations denoted simply by the juxtaposition of a scalar and a vector—but trying to solve this problem leads either to introducting some symbol for scalar multiplication (for which both "·" and "×" could be confusing), which we would then modify with a subscript as in (3), or to replacing (2) with the opaque-looking

$$\mathrm{sm}_{V\oplus_e W}(c,(v,w)) := \big(\mathrm{sm}_V(c,v), \mathrm{sm}_W(c,w)\big).$$

In this instance, most mathematicians deem the cure to be worse than the disease, and tolerate the"abuse of notation" in equations (1) and (2) as the lesser evil.

Exercise DS1. Compare the vector spaces $\mathbf{R} \oplus_e \mathbf{R}$ and $\mathbf{R}^2$ (where the latter is equipped with its standard vector-space structure).

"Internal" direct sum

Let Z be a vector space and let V and W' be subspaces of Z. Then the *subspace sum* $V + W'$ is also a subspace of Z, as shown in another homework exericise (Friedberg, Insel, and Spence, 5th edition, exercise 1.3/ 23). Two conditions on V and W', each of which can (in general) be satisfied or failed independently of the other, are (i) $V + W' = Z$ and (ii) $V \cap W' = 0_{\mathbf{Z}}$.

If *both* (i) and (ii) are satisfied, we say that W' is a *complement*[4] of V (in Z), or that V is a complement of W', or that V and W' are *complements* or *complementary subspaces*. In this case we also say that Z is the "internal" direct sum of V and W', for which we are using the notation $V \oplus_i W'$ in these notes.

The next exercise shows that every external direct sum can be expressed as an internal direct sum, and vice-versa.

Exercise DS2. Let V and W be vector spaces. Define subsets V', W' of $V \times W$ by $V' = V \times \{0_W\} = \{(v, 0_W) : v \in V\}$ and $W' = \{0_V\} \times W = \{(0_V, w) : w \in W\}$.

(a) Show that V' and W' are subspaces of $V \oplus_e W$.

(b) Show that these two subspaces of $V \oplus_e W$ are complements of each other, and hence that $V \oplus_e W = V' \oplus_i W'$.

[4]This linear-algebraic meaning of *complement* is very different from the set-theoretic meaning.

Later in the semester, when we define what it means for two vector spaces to be *isomorphic*, we will revisit the exercise above (perhaps just in homework) and see the subspaces V' and W' are isomorphic to the vector spaces V and W, respectively. This strengthens the tie between external and internal direct sums, showing them to be "almost the same thing" viewed two different ways.

Although the two types of direct sum are *almost* the same, they are not *quite* the same, for the following reason. If V' and W' are complementary subspaces of a vector space Z', then $Z' = V' \oplus_i W' = W' \oplus_i V$. However, if V and W are two vector spaces and $V \neq W$, then $V \oplus_e W$ is not literally the same vector space as $W \oplus_e V$; for the first of these, the set of elements of the direct sum is $V \times W$, whereas for the other the set of elements is $W \times V$. Of course, there is a natural bijection from $V \times W$ to $W \times V$ that sends (v, w) to (w, v), so any statement we can make about $V \oplus_e W$ can be translated into an equivalent statement about $W \oplus_e V$. (When we revisit direct sums after we define isomorphism, we'll see that this "swapping" map is an isomorphism, and hence that $V \oplus W$ and $W \oplus V$ are isomorphic.)

Proposition DS1 ("unique decomposition" property of direct sums). *Let V and W be complementary subspaces of a vector space Z. Then for every $z \in Z$, there exist unique vectors $v \in V$ and $w \in W$ such that $z = v + w$.*

(In other words, every $z \in Z$ can be *uniquely expressed* as an element of V plus an element of W.)

Exercise DS3. Prove Proposition DS1.