Clarification of item in Friday, September 15 lecture

In class on Friday I gave two definitions (which I called "Definition 1" and "Definition 1′") of "linearly dependent set" and "linearly independent set", and later showed that Definitions 1 and 1′ are equivalent. In response to a question, shortly after writing Definition 1′1 on the left-hand blackboard, I added (on the same blackboard) what I said was an alternative wording. However, that "alternative wording" is more precise than my first wording, as I'll discuss below. Here are Definitions 1 and 1′, using the more precise wording of Definition 1′:

Definition 1. Let V be a vector space and let $S \subseteq V$.

(a) We say that S is *linearly dependent* if there exist an integer $n \geq 1$, and distinct vectors $v_1, \ldots, v_n \in S$ (where "distinct" means that $v_i \neq v_j$ whenever $i \neq j$) , and scalars $c_1, \ldots, c_n$ that are not all zero, such that

$$c_1 v_1 + \cdots + c_n v_n = 0_V \ .$$

(b) We say that S is *linearly independent* if S is not linearly dependent.

Definition 1′. Let V be a vector space and let $S \subseteq V$.

(a) We say that S is *linearly dependent* if there exists a vector $v \in S$ such that $v \in \mathrm{span}(S \backslash \{v\})$.

(b) We say that S is *linearly independent* if S is not linearly dependent.

If S is a singleton set $\{v\}$ (i.e. if S has only one element, which we're calling v), then $\mathrm{span}(S \setminus \{v\}) = \{0_V\}$, and hence

$$v \in \mathrm{span}(S \backslash \{v\} \ \text{ if and only if } \ v = \{0_V\}.$$

Thus, according to Definition 1′, the set $\{v\}$ is linearly independent if $v \neq 0_V$, but the set $\{0_V\}$ is linearly dependent. Observe that this is consistent with Definition 1.

My original wording of part (a) of Definition 1′ was: "We say that S is *linearly dependent* if there exists a vector $v \in S$ such that v is a linear combination of other elements of S." The problem with that wording is that it can be misleading when S consists *only* of the zero vector. If S has only one element, say v, then S has no *other* elements, so it would appear that v cannot be a "linear combination of other elements of S". However, we've defined $\mathrm{span}(\emptyset)$ to be the singleton set $\{0_V\}$, and in order to have the definition

$$\mathrm{span}(S) = \{\text{all linear combinations of elements of } S\}$$

give what we want it to for *every* subset S of V, we defined the term "linear combination of elements of the empty subset of V" to mean the zero vector, 0_V. In order to ensure that the original wording of Definition $1'$ is equivalent to Definition 1, we have to interpret "linear combination of other elements of S" as meaning "linear combination of elements of $S \backslash \{v\}$."