<u>Non-book problems for Assignment 11</u>

[Assorted problems that I considered putting on the Nov. 8 exam. Some are answered in book.]

NB 11.1. Let $n \geq 1$ and let $B, C \in M_{n\times n}(\mathbf{R})$. Define functions $\tilde{L}_B, R_C : M_{n\times n}(\mathbf{R}) \to M_{n\times n}(\mathbf{R})$ by $\tilde{L}_B(A) = BA$ and $R_C(A) = AC$. (I'm using the notation "$\tilde{L}_B$" rather than "L_B" since we've been using the latter for a specific map from $\mathbf{R}^n$ to $\mathbf{R}$.)

(a) Show that $\tilde{L}_B$ and R_C are linear.

(b) Show that $\tilde{L}_B \circ R_C = R_C \circ \tilde{L}_B$.

(c) Let β be a basis of $M_{n\times n}(\mathbf{R})$. What size are the matrices $[\tilde{L}_B]_\beta$ and $[R_C]_\beta$? (I.e., what are the numbers r and s for which these matrices are $r \times s$ matrices?)

(d) Is it always true that $BC = CB$? If yes, prove it; if not, give a counterexample with $n = 2$.

(e) Is it always true that $[\tilde{L}_B]_\beta\ [R_C]_\beta = [R_C]_\beta[\tilde{L}_B]_\beta$? (You should be able to deduce the answer from earlier part(s) of this problem.) If yes, prove it; if not, give a counterexample with $n = 2$.

NB 11.2. Let V be a finite-dimensional vector space and let $\mathsf{T} : \mathsf{V} \to \mathsf{V}$ be linear. Show that if $\mathsf{T}^2 = \mathsf{T}$, then (a) $\mathsf{V} = \mathsf{N}(\mathsf{T}) \oplus \mathsf{R}(\mathsf{T})$, and (b) T is the projection on $\mathsf{R}(\mathsf{T})$ along $\mathsf{N}(\mathsf{T})$.

Suggestion for part (a): Use the fact that $v = \big(v - \mathsf{T}(v)\big) + \mathsf{T}(v)$ (for every $v \in \mathsf{V}$).

NB 11.3. [Check whether this is in book.]

Let $V = P_2(\mathbf{R})$ (the space of polynomials of degree at most 2).

(a) Define $L : V \to V$ by $L(f)(x) = f(x - 1)$. Show that L is linear.

(b) Find the matrix of L with respect to the standard ordered basis $\{1, x, x^2\}$ of V.

NB 11.4. Show that the inverse of an invertible LT is linear.

NB 11.5. Give an example of two distinct infinite-dimensional vector spaces V, W that are isomorphic to each other. Also, give an explicitly defined map $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ that is an isomorphism. (You're not being asked to *prove* that your map T is an isomorphism, or that it has any of the properties an isomorphism must have. You're just being asked to write down a map $\mathsf{T} : \mathsf{V} \to \mathsf{W}$ that, indeed, <u>is</u> an isomorphism.)

(? points) [S92 final] Let V be the space of symmetric 2x2 matrices and W the space of antisymmetric 3x3 matrices. (Recall that a matrix is *symmetric* if it is equal to its transpose, and *antisymmetric* if it is equal to minus its transpose.) Find an isomorphism from V to W. (You may find it useful to write down what general elements of V and W look like.)

NB 11.6. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation whose matrix in the standard ordered basis β is $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$. Let γ be the ordered basis $\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$.

(a) Find the matrix B that changes γ-coordinates to β-coordinates.

(b) Find the matrix C that changes β-coordinates to γ-coordinates.

(c) DO NOT DO ANY COMPUTATION IN THIS PART. In terms of A, B and C, find the matrix of T in the ordered basis γ. (I.e. write down the matrix product that you would have to compute to get the numerical entries of $[T]_\gamma$, but DO NOT perform the matrix multiplication.)

NB 11.7. Suppose A is an $M \times M$ matrix for which $A^n = 0$ for some $n > 1$. Show that

$$I + A + A^2 + \cdots + A^{n-1} = (I - A)^{-1}.$$