Non-book problems for Assignment 12

Notation and terminology to recall: For any set S, we define the notation $S \times S$ by

$$\begin{aligned} S \times S &:= \{(s_1, s_2) : s_1, s_2 \in S\} \\ &= \text{the Cartesian product of } S \text{ with itself} \\ &= \text{the set of ordered pairs of elements of } S \\ &= \text{the set of two-term lists of elements of } S. \end{aligned}$$

Similarly, for any $n \geq 2$, we define

$$\begin{aligned} \underbrace{S \times S \times \cdots \times S}_{k \text{ copies of } S} &:= \{(s_1, s_2, \ldots, s_k) : s_1, s_2, \ldots, s_k \in S\} \\ &= \text{the } k\text{-fold Cartesian product of } S \text{ with itself} \\ &= \text{the set of ordered } k\text{-tuples of elements of } S \\ &= \text{the set of } k\text{-term lists of elements of } S. \end{aligned}$$

(The notation doesn't have to include anything underneath "$S \times S \times \cdots \times S$" if the number of copies of S is clear from context.)

NB 12.1. Let V and W be vector spaces. A function $f : V \times V \to W$ is called *bilinear* if f is linear as a function of each variable with the other variable held fixed; i.e. if, for every fixed $z \in V$, each of the two functions from V to W defined by $v \mapsto f(v, z)$ and $v \mapsto f(z, v)$ is linear. Equivalently, f is bilinear if l $v_1, v_2, u \in V$ and $c \in \mathbf{R}$,

$$\begin{aligned} f(v_1 + cu, v_2) &= f(v_1, v_2) + cf(u, v_2) \\ &\text{and} \\ f(v_1, v_2 + cu) &= f(v_1, v_2) + cf(v_1, u). \end{aligned}$$

A function $f : V \times V \to W$ is called *alternating* (or *antisymmetric*) if $f(v_1, v_2) = -f(v_2, v_1)$ for all $v_1, v_2 \in V$.

(a) Let $f : V \times V \to W$ be a bilinear function. Show that

$$f(v_1 + v_2, v_3 + v_4) = f(v_1, v_3) + f(v_1, v_4) + f(v_2, v_3) + f(v_2, v_4).$$

(b) Again let $f : V \times V \to W$ be a bilinear function. Show that f is alternating if and only if $f(v, v) = 0_W$ for all $v \in V$.

Hint for the "if" part: Consider $f(v_1 + v_2, v_1 + v_2)$ for arbitrary $v_1, v_2 \in V$.

NB 12.2. We can extend the concepts and result in the previous problem extend to functions of more than two variables. For any $k \geq 2$ and any vector spaces V and W, a function $f : \underbrace{V \times V \times \cdots \times V}_{k \text{ copies of } V} \to W$ is called *multilinear* if f is linear as a function of each variable with all the other variables held fixed. (When $k = 2$, *multilinear* means *bilinear*; when $k = 3$, we often say *trilinear* instead of *multilinear*.) E.g. a function $f : V \times V \times V \to W$ is multilinear (or trilinear) if for all $z_1, z_2 \in V$, each of the following is a linear function from V to W:

$$\begin{aligned} v &\mapsto f(v, z_1, z_2), \\ v &\mapsto f(z_1, v, z_2), \\ \text{and} \qquad v &\mapsto f(z_1, z_2, v). \end{aligned}$$

A function $f : \underbrace{V \times V \times \cdots \times V}_{k \text{ copies of } V} \to W$ is called *alternating* (or *totally antisymmetric*, with "totally" omitted if $k = 2$) if, whenever any $k-2$ of the variables are held fixed, f is alternating as a function of the remaining two variables. For example, $f : V \times V \times V \to W$ is alternating if for all $v_1, v_2, v_3 \in V$,

$$\begin{aligned} & f(v_1, v_2, v_3) = -f(v_2, v_1, v_3) \quad \text{[the 1}^{\text{st}}\text{ and 2}^{\text{nd}}\text{ variables have been interchanged]}, \\ & f(v_1, v_2, v_3) = -f(v_3, v_2, v_1) \quad \text{[the 1}^{\text{st}}\text{ and 3}^{\text{rd}}\text{ variables have been interchanged]}, \\ \text{and} \quad & f(v_1, v_2, v_3) = -f(v_1, v_3, v_2) \quad \text{[the 2}^{\text{nd}}\text{ and 3}^{\text{rd}}\text{ variables have been interchanged]}. \end{aligned}$$

(a) Show that a multilinear function $f : \underbrace{V \times V \times \cdots \times V}_{k \text{ copies of } V} \to W$ is alternating if, for all $v_1, v_2, \ldots, v_k \in V$,

$$f(v_1, \ldots, v_k) = 0_W \quad \text{if } v_i = v_j \text{ for some distinct } i, j \in \{1, \ldots, k\}. \tag{1}$$

(For example, with $k = 3$, statement (1) says that for all $v, u \in V$, $f(v, v, u) = 0_W = f(v, u, v) = f(u, v, v)$.)

(b) Recall from Calculus 3 (or physics) that the *cross product* is a function from $\mathbf{R}^3 \times \mathbf{R}^3 \to \mathbf{R}^3$ whose definition can be written as

$$(a_1, a_2, a_3) \times (b_1, b_3, b_3) = \left(\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, -\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right).$$

Verify that cross-product is a bilinear, alternating function.

NB 12.3. Let $n \geq 2$. Viewing elements of $\mathbf{R}^n$ as row-vectors, define $f : \underbrace{\mathbf{R}^n \times \mathbf{R}^n \times \cdots \times \mathbf{R}^n}_{n \text{ copies}} \to \mathbf{R}$ by

$$f(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n) = \det \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix}.$$

Check that, in chapter 4 of FIS, Theorems 4.3 and 4.5 together state that f is a multlinear, alternating function.

This fact is often stated, more informally, as: determinant of $n \times n$ matrices is an alternating, multilinear function of the rows.

Note: The corresponding statement using the *columns* of an $n \times n$ matrix, instead of the rows, is also true. Although the statements for rows and for columns are both true, the statement for columns is the more common of the two, simply because treating elements of $\mathbf{R}^n$ as *column* vectors is more convenient for matrix algebra. I've stated this homework problem using rows just because that's our book's approach.