<u>**Non-book problems for Assignment 15**</u>

**In the problems below, $(V, \langle\ ,\ \rangle)$ is a finite-dimensional inner-product space of positive dimension $n$.**

**NB 15.1**. Let $\beta = \{u_1, \ldots, u_n\}$ be an orthonormal basis of $V$, and let $v, w \in V$. Let $\vec{a} = [v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\vec{b} = [w]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. (Thus $v = \sum_i a_i u_i$ and $w = \sum_i b_i u_i$.)

(a) Show that

$$\langle v, w \rangle = \sum_{i=1}^{n} a_i b_i = [v]_\beta \cdot [w]_\beta. \tag{1}$$

In other words, in an orthonormal basis of $(V, \langle\ ,\ \rangle)$, the inner product "looks just like" dot-product. (**This is NOT true in an *arbitrary* basis!!**)

(b) From equation (3), deduce that if $a_1, \ldots, a_n$ are the coordinates of a vector $v \in V$ with respect to an orthonormal basis,

$$\|v\|^2 = {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2\ . \tag{2}$$

(**Again, this is NOT true for an *arbitrary* basis!!**)

**NB 15.2**. Let $v_1, \ldots, v_n$ be a list of vectors in $\mathbf{R}^n$ (written as column vectors), and let $A \in M_{n\times n}(\mathbf{R})$ be the matrix whose $i^{\text{th}}$ column is $v_i$. Equip $\mathbf{R}^n$ with the standard inner product. Show that $\{v_1, \ldots, v_n\}$ is an orthonormal basis of $\mathbf{R}^n$ if and only if $A^t A = I$ (the identity $n \times n$ matrix).

**NB 15.3.** Let $W$ be a subspace of $V$. In class, we showed that $V = W \oplus W^\perp$, and that if $m = \dim(W) > 0$, then the map $\text{proj}_W : V \to V$ (the projection map onto $W$ along $W^\perp$, also called the *orthogonal projection* from $V$ to $W$), can be comoputed from any orthonormal basis $\{w_1, \ldots, w_n\}$ of $W$ by the formula

$$\text{proj}_W(v) = \sum_{i=1}^{m} \langle v, w_i \rangle w_i$$

(a) Show that, for all $v \in V$,

$$\text{proj}_W(v) + \text{proj}_{W^\perp}(v) = v. \tag{3}$$

Equivalently, the maps $\text{proj}_W, \text{proj}_{W^\perp}$ in the vector space $\mathcal{L}(V, V)$ satisfy

$$\text{proj}_W + \text{proj}_{W^\perp} = I_V\ . \tag{4}$$

(b) Do the analogs of equation (3) and (4) hold for *any* direct-sum decomposition of an arbitrary vector space $V'$ as $W_1 \oplus W_2$, where $W_1, W_2$ are complementary subspaces of $V'$), or is it special to finite-dimensional inner-product spaces and orthgonal complements?

**NB 15.4** Consider the case in which $V = \mathbf{R}^n$, where $n \geq 2$, and $\langle\ ,\ \rangle$ is the standard inner product on $\mathbf{R}^n$.

For $k \in \{1, \ldots, n-1\}$ let

$$\begin{aligned} H_k &= \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \in \mathbf{R}^n : a_i = 0 \text{ for all } i > k \right\} \\ &= \left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} : a_1, \ldots, a_k \in \mathbf{R} \right\} \\ &= \text{span}\{\vec{e}_1, \ldots, \vec{e}_k\}. \end{aligned}$$

(a) Explicitly, what is the subspace $(H_k)^\perp$?

(b) Let $\vec{a} = (a_1, \ldots, a_n)$ and let $k \in \{1, 2, \ldots, n-1\}$. Compute $\text{proj}_{H_k}(\vec{a})$ and $\text{proj}_{(H_k)^\perp}(\vec{a})$ in terms of $a_1, \ldots a_n$. (Once you find the first of these, you should be able to write down the the second immediately, without any real computations, from earlier problem(s) in this assignment.)

**NB 15.5** Let

$$A = \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{pmatrix} \in M_{n \times n}(\mathbf{R}).$$

Consider the associated system of $m$ homogeneous linear equations in $n$ (scalar) unknowns, written in vector form as

$$A\vec{x} = \vec{0} \quad (= \vec{0}_{\mathbf{R}^m}), \tag{5}$$

with vector variable $\vec{x} \in \mathbf{R}^n$. Observe, or recall, that the solution-space of (5) is precisely $\mathsf{N}(A)$ (the null space of $A$, which by definition is the null space of the linear map $L_A : \mathbf{R}^n \to \mathbf{R}^m$).

Below, equip $\mathbf{R}^n$ with its standard inner product.

(a) Show that

$$\mathsf{N}(A) = (\text{row-space of } A)^{\perp}. \tag{6}$$

(b) Let $W$ be an $m$-dimensional subspace of $\mathbf{R}^n$ where $m > 0$. In class we showed that $W^{\perp}$ satisfies the de

nition of "a complement of $W$" introduced early this semester. Without using the knowledge that $W^{\perp}$ is a complement of $W$, use relation (6), together with the Rank-Plus-Nullity Theorem, to give another proof that

$$\dim(W^{\perp}) = n - \dim(W).$$