Non-book problems for Assignment 9

NB 9.1. Let V, W, Z be vector spaces, with V and W finite-dimensional. Let $T : V \to W$ and $S : W \to Z$ be linear transformations. Show that

$$\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}.$$

Before starting, prove the following trivial lemma for yourself: for any real numbers x, y, z, the inequality "$x \leq \min\{y, z\}$" is equivalent to "$x \leq y$ and $x \leq z$." After proving this lemma for yourself, don't bother citing it when you use it. Thus, for example, to show $\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}$ in the exercise above, you should show that $\text{rank}(S \circ T) \leq \text{rank}(S)$ and that $\text{rank}(S \circ T) \leq \text{rank}(T)$, and then say something like "Therefore $\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}$;" you should not break the argument down into cases according to which of $\text{rank}(S)$ and $\text{rank}(T)$ is the larger.

Something you may find helpful: for any vector spaces $\tilde{W}$ and $\tilde{Z}$, any subspace $X \subseteq \tilde{W}$, and any linear map $\tilde{S} : \tilde{W} \to \tilde{Z}$, the *restriction of* $\tilde{S}$ *to* X—i.e. the map $S|_X : X \to \tilde{Z}$ defined by $\tilde{S}|_X(x) = \tilde{S}(x)$ for all $x \in X$—is a linear map from X to $\tilde{Z}$ (why?).

NB 9.2. (We did some or all parts of this problem in class a few weeks ago, but I don't think I displayed the results in numbered proposition or corollaries. In this problem, do any such proofs over again; don't just say "This is true because we proved it in class.")

Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear map.

(a) Show that

$$\text{rank}(T) \leq \min\{\dim(V), \dim(W)\}. \tag{1}$$

(We actually showed this in class, briefly, a few weeks ago, but I don't think I displayed it as a numbered proposition or corollary. Although you *could* derive this from problem NB 9.1 by doing something clever, don't try; it's not worth the effort. The inequality (1) holds for simpler reasons.)

(a) Show that T is one-to-one if and only if $\text{rank}(T) = \dim(V)$.

(b) Show that T is onto if and only if $\text{rank}(T) = \dim(W)$.

NB 9.3. Let V and W be finite-dimensional vector spaces and assume that $\dim(V) < \dim(W)$.

(a) Show that there exists a one-to-one linear transformation from V to W. (*Hint*: Theorem 2.6 in FIS Chapter 2.)

(b) Show that if $T : V \to W$ is a one-to-one linear transformation, then there exists a linear transformation $S : W \to V$ such that $S \circ T = I_V$.

(c) Show that if $S : W \to V$ is a linear transformation that is onto, then there exists a linear transformation $T : V \to W$ such that $S \circ T = I_V$. Show also that any such T is one-to-one. (*Note*: The S and T in this problem-part are *new*; they're not carried over from part (b). In fact, **the wording of part (b) tells you this. How?**)

(d) Give two proofs, as indicated below, that there do not exist any linear transformations $T : V \to W$ and $S : W \to V$ for which $T \circ S = I_W$:

- *Hint for Proof #1*: Use Problem NB 9.1.
- *Hint for Proof #2*: Show that for any sets X, Y and functions $f : X \to Y$ and $g : Y \to X$, if $g \circ f = I_X$ (the identity map of X), then f is one-to-one and g is onto. Then use what we proved several weeks ago about non-existence of certain linear maps from V to W, and certain linear maps from W to V, under the given assumption that $\dim(V) < \dim(W)$.

NB 9.4. Let V and W be finite-dimensional vector spaces of equal dimension n. Suppose that $T : V \to W$ and $S : W \to V$ are linear transformations for which $S \circ T = I_V$. Using the steps below, prove that $T \circ S = I_W$.

(a) Show that $\text{rank}(S) = \text{rank}(T) = n$. (*Hint*: Problem NB 9.1.) In particular, T is onto.

(b) Show that $T \circ S \circ T = T$. (Recall that composition of composable functions is associative: if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$. Hence both sides of the latter equation can unambiguously be denoted $h \circ g \circ f$.)

(c) Now use parts (a) and (b) to show that $T \circ S = I_W$.

Note: Problem NB 9.3(d) shows that **in order for "$S \circ T = I_V$" to imply "$T \circ S = I_W$" non-vacuously, the assumption that $\dim(V) = \dim(W)$ is *crucial* for !** (The "non-vacuously" means here that it's *possible* for the condition "$S \circ T = I_V$" to be met.)

NB 9.5. Let m, n be positive integers with $n < m$.

(a) Show that if $A \in M_{m \times n}(\mathbf{R})$ is such that the map $L_A : \mathbf{R}^n \to \mathbf{R}^m$ has rank n, then there exists a matrix $B \in M_{n \times m}(\mathbf{R})$ such that $BA = I_{n \times n}$.

(*Hint*: Use appropriate parts of Problem NB 9.2 and NB 9.3.)

(b) Show that there are no matrices $A \in M_{m\times n}(\mathbf{R}), B \in M_{n\times m}(\mathbf{R})$ such that $AB = I_{m\times m}$.

(Same hint.)

NB 9.6. Let $A, B \in M_{n\times n}(\mathbf{R})$ and let $I = I_{n\times n}$. Show that if $AB = I$, then $BA = I$. (*Hint*: Problem NB 9.4.)

NB 9.7. Let $m, n, p \in \mathbf{N}$.

(a) Let $A \in M_{m\times n}(\mathbf{R})$ and $B \in M_{n\times m}(\mathbf{R})$. Then both the products AB and BA are defined, and both are square matrices (the first is $m \times m$; the second is $n \times n$), so the trace of each is defined. Show that

$$\mathrm{tr}(AB) = \mathrm{tr}(BA), \tag{2}$$

whether or not $m = n$. (This generalizes the first part of FIS exercise 2.3/13, in which you showed that equation (2) holds if both A and B are $n \times n$.)

(b) Let $A \in M_{m\times n}(\mathbf{R}), B \in M_{n\times p}(\mathbf{R})$, and $C \in M_{p\times m}(\mathbf{R})$. Check that each of the products ABC, BCA, and CAB is defined and is a square matrix, and show that

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB). \tag{3}$$

(Use the *result of* part (a) to do this very quickly; don't give a lengthier version of the *argument* you used for part (a).)

Observe that the permutations of "A, B, C" appearing in equation (4) are only the *cyclic permutations*, not *all* permutations. Equation (4) (as well as its generalization in part (c)), is often called the "cyclic property of the trace."

(c) Formulate and prove a generalization of part (b) for arbitrarily many appropriately-sized matrices.

NB 9.8. Let A and B be diagonal $n \times n$ matrices. Show that AB is also a diagonal matrix, and that if the diagonal entries of A are $\lambda_1, \dots, \lambda_n$ and the diagonal entries of B are $\mu_1, \dots, \mu_n$ (i.e. $\lambda_i = A_{ii}$ and $\mu_i = B_{ii}$, $1 \le i \le n$), then the diagonal entries of AB are simply the products $\lambda_1\mu_1, \dots, \lambda_n\mu_n$.

NB 9.9. (a) Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Compute A^2, A^3, and A^4.

(b) In part (a), you should have found that A^4 is a very simple matrix—so simple that you can immediately tell what all higher powers of A would be. You should also have noticed a pattern in location of the nonzero above-diagonal entries in the sequence A, A^2, A^3. Does the value of A^4, or the pattern you noticed in the sequence A, A^2, A^3, depend at all on the values of the above-diagonal entries of A?

(c) For $n \times n$ matrices with $n \geq 2$, conjecture how your observations in part (b) would generalize.

(d) Try to prove the conjecture you made in part (c).

NB 9.10. Let $A, C \in M_{n\times n}(\mathbf{R})$ and assume that C is invertible. Show that for any integer $k \geq 1$,

$$(C^{-1}AC)^k = C^{-1}A^k\, C$$

and similarly

$$(CAC^{-1})^k = C\, A^k\, C^{-1}.$$

NB 9.11. *Matrix model for the complex number system.* If you need to review complex numbers before doing this problem, see Appendix D in FIS.

(a) Check that $\mathbf{C}$, the space of complex numbers, is a real vector space for which $\{1, i\}$ is a basis. (Hence the dimension of this real vector space is two.)

(b) In $M_{2\times 2}(\mathbf{R})$, let

$$I = I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and let $H = \text{span}\{I, J\} \subseteq M_{2\times 2}(\mathbf{R})$. Clearly $\{I, J\}$ is a linearly independent set, so H is a two-dimensional (real) vector space for which $\{I, J\}$ is a basis.

Check that H is the space of real matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

(c) Compute J^2 and express your answer in terms of I.

(d) Show that H is closed under multiplication (of 2×2 matrices). I.e. show that if $Z, W \in H$, then $ZW \in H$.

(e) Define $\phi : H \to \mathbf{C}$ to be the linear map for which $\phi(I) = 1$ and $\phi(J) = i$. (Thus $\phi(aI + bJ) = a + bi$ for all $a, b \in \mathbf{R}$.) Show ϕ is an isomorphism (of vector spaces) and that, in addition,

$$\phi(ZW) \quad = \quad \phi(Z)\phi(W) \quad \text{for all } Z, W \in H. \tag{4}$$

(The right-hand side of equation (4) is the product of the complex numbers $\phi(Z)$ and $\phi(W)$.)

Thus $\phi : H \to \mathbf{C}$ is a bijective (i.e. one-to-one and onto) map that carries matrix addition (of matrices in H) to addition of complex numbers, and carries matrix multiplication (of matrices in H) to multipliication of complex numbers.

(f) In the usual introduction to complex numbers, we define the multiplication operation by declaring the product $(a + bi)(c + di)$ to be $ac - bd + (ad + bc)i$ (where $a, b, c, d \in \mathbf{R}$). The question then arises: is this operation associative? To verify that it *is* associative, we usually then take three arbitrary complex numbers, say $z_1 = a + bi, z_2 = c + di,\ z_3 = e + fi$ (where $a, b, c, d, e, f \in \mathbf{R}$), compute $(z_1z_2)z_3$ and $z_1(z_2z_3)$, and check that the results are equal. Give a *different* proof of the associativity of complex multiplication, using the map ϕ and the associativity of matrix multiplication.