Using elementary operations to compute determinants

Below, except in examples with specific matrices, A and B are arbitrary $n \times n$ matrices, where $n \geq 1$ is fixed but arbitrary and $c \in \mathbf{R}$ is an arbitrary scalar.

Elementary row/column operations have simple effects on the determinant, because of the *alternating* and *multilinearity* properties of the determinant function on $M_{n\times n}(\mathbf{R})$:

- Type-1 operations change the determinant by a sign.

  I.e. if P is an operation of the type I've denoted Rop_{1ij}, then $\det(P(A)) = -\det(A)$.

- The type-2 operations I've denoted Rop_{2ic} and Cop_{2ic} change the determinant by a factor of c.

  I.e., if P is either of these operations, then $\det(P(A)) = c\det(A)$.

  When used in equations involving determinants, I'll refer to this as "factoring c out of row/column i."

- Type-3 operations do not change the determinant at all: if P is one of these operations, then $\det(P(A)) = \det(A)$.

  For example, if the columns of A are $v_1, \ldots, v_n$, and P is the operation I've denoted Cop_{3ijc}, with $i = 1$ and $j = 2$ for concreteness' sake,

$$\begin{aligned}
\det(P(A)) &= \det\left(\, v_1 + cv_2 \mid v_2 \mid \ldots \mid v_n \, \right) \qquad \text{(by def. of } \mathrm{Rop}_{3,12c}\text{)} \\
&= \det\left(\, v_1 \mid v_2 \mid \ldots \mid v_n \, \right) + \det\left(\, cv_2 \mid v_2 \mid \ldots \mid v_n \, \right) \\
&\qquad \text{[by linearity of the determinant in the first column separately]} \\
&= \det(A) + 0 \\
&\qquad \text{[since, in the second determinant, column 1 is a multiple of column 2]} \\
&= \det(A).
\end{aligned}$$

Together, the first two bullet-points imply another important property of determinants:

- We will be using the fact that if one column of A is a scalar multiple of another column, or one row of A is a scalar multiple of another row, then $\det(A) = 0$.

Expanding a determinant along a row or column is often the *least* efficient way to compute the determinant. When a matrix has a lot of nonzero entries, the facts above can be used to simplify the computation considerably, and—for *some* determinants involving variable entries—to help us factor the determinant as a polynomial in these variables.

(Factoring a determinant called the *characteristic polynomial* is the most common way to find *eigenvalues*, a Chapter 5 topic. Example 4, later, is an instance in which elementary operations help with this factorization.)

Example 1

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} \underset{R_3 \to R_3 - R_2}{=} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 3 & 3 \end{vmatrix} \underset{R_2 \to R_2 - R_1}{=} \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{vmatrix}$$

$$= \quad 0 \quad \text{(since two rows are identical).}$$

Example 2

$$\begin{aligned}
\begin{vmatrix} 13 & 14 & 15 \\ 14 & 15 & 13 \\ 15 & 13 & 14 \end{vmatrix} &\underset{\substack{R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - R_1}}{=} \begin{vmatrix} 13 & 14 & 15 \\ 1 & 1 & -2 \\ 2 & -1 & -1 \end{vmatrix} \\
&\underset{\substack{R_1 \longrightarrow R_1 - 13R_2 \\ R_3 \longrightarrow R_3 - 2R_2}}{=} \begin{vmatrix} 0 & 1 & 41 \\ 1 & 1 & -2 \\ 0 & -3 & 3 \end{vmatrix} \\
&= 0 \cdot \text{something} - 1 \cdot \begin{vmatrix} 1 & 41 \\ -3 & 3 \end{vmatrix} + 0 \cdot \text{something} \\
&\qquad \text{(expanding previous determinant along 1st column)} \\
&= -(3 + 123) \\
&= -126,
\end{aligned}$$

less painfully than expanding the original determinant along any row or column.

Example 3 **(a)** Let x, y, and z be arbitrary real numbers. Then

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} \underset{R_3 \to R_3 - x\,R_2}{=} \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 0 & y^2 - xy & z^2 - xz \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ 0 & y(y-x) & z(z-x) \end{vmatrix}$$

$$\underset{R_2 \to R_2 - x\,R_1}{=} \begin{vmatrix} 1 & 1 & 1 \\ 0 & y-x & z-x \\ 0 & y(y-x) & z(z-x) \end{vmatrix}$$

$$= 1 \cdot \begin{vmatrix} y-x & z-x \\ y(y-x) & z(z-x) \end{vmatrix} - 0 \cdot \text{something} + 0 \cdot \text{something}$$

(expanding previous determinant along 1st column)

$$= (y-x)(z-x) \begin{vmatrix} 1 & 1 \\ y & z \end{vmatrix}$$

(factoring $y - x$ out of column 1,
then factoring $z - x$ out of column 2)

$$= (y-x)(z-x)(z-y).$$

(b) Let x_1, x_2, x_3, x_4 be real numbers. Then, by a procedure analogous to the one in part (a), and with notation like "${x_3}^2$" meaning "$(x_3)^2$", we find that

$$
\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 \\ {x_1}^3 & {x_2}^2 & {x_3}^3 & {x_4}^3 \end{vmatrix}
\underset{R_4 \to R_4 - x_1 R_3}{=}
\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ {x_1}^2 & {x_2}^2 & {x_3}^2 & {x_4}^2 \\ 0 & {x_2}^3 - x_1 {x_2}^2 & {x_3}^3 - x_1 {x_3}^2 & {x_4}^3 - x_1 {x_4}^2 \end{vmatrix}
$$

$$
\underset{R_3 \to R_3 - x_1 R_2}{=}
\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ 0 & {x_2}^2 - x_1 x_2 & {x_3}^2 - x_1 x_3 & {x_4}^2 - x_1 x_4 \\ 0 & {x_2}^3 - x_1 {x_2}^2 & {x_3}^3 - x_1 {x_3}^2 & {x_4}^3 - x_1 {x_4}^2 \end{vmatrix}
$$

$$
\underset{R_2 \to R_2 - x_1 R_1}{=}
\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ 0 & {x_2}^2 - x_1 x_2 & {x_3}^2 - x_1 x_3 & {x_4}^2 - x_1 x_4 \\ 0 & {x_2}^3 - x_1 {x_2}^2 & {x_3}^3 - x_1 {x_3}^2 & {x_4}^3 - x_1 {x_4}^2 \end{vmatrix}
$$

$$
= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ {x_2}^2 - x_1 x_2 & {x_3}^2 - x_1 x_3 & {x_4}^2 - x_1 x_4 \\ {x_2}^3 - x_1 {x_2}^2 & {x_3}^3 - x_1 {x_3}^2 & {x_4}^3 - x_1 {x_4}^2 \end{vmatrix}
$$

(expanding previous determinant along 1st column)

$$
= \begin{vmatrix} x_2 - x_1 & x_3 - x_1 & x_4 - x_1 \\ x_2(x_2 - x_1) & x_3(x_3 - x_1) & x_4(x_4 - x_1) \\ {x_2}^2(x_2 - x_1) & {x_3}^2(x_3 - x_1) & {x_4}^2(x_4 - x_1) \end{vmatrix}
$$

$$
= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1) \begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ {x_2}^2 & {x_3}^2 & {x_4}^2 \end{vmatrix}
$$

(factoring $x_2 - x_1$ out of column 1,
then factoring $x_3 - x_1$ out of column 2)
then factoring $x_4 - x_1$ out of column 3)

$$
= (x_2 - x_1)(x_3 - x_1)(x_4 - x_1)(x_3 - x_2)(x_4 - x_2)(x_4 - x_3)
$$

(using part (a)).

$$
= \prod_{1 \le i < j \le 4} (x_j - x_i),
$$

the product of all expressions $x_j - x_i$ with $1 \le i < j \le 4$. ("Π" is used for products the way "Σ" is used for sums.)

The strategy above can be used to create an inductive proof that, for any $n \ge 2$, and

$x_1, \ldots, x_n \in \mathbf{R}$ (or in $\mathbf{C}$),

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_n \\ {x_1}^2 & {x_2}^2 & \ldots & {x_n}^2 \\ \vdots & \vdots & \vdots & \vdots \\ {x_1}^{n-1} & {x_2}^{n-1} & \vdots & {x_n}^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i). \tag{1}$$

Fun fact, just FYI. Equation (1) is a famous formula with many uses, and can be proven a few different, ways. It is also a an *elegant* formula. From the definition of determinant, it is easy to see that the left-hand side of (1) is a polynomial in the n variables $x_1, \ldots x_n$. Furthermore, this polynomial function must evaluate to 0 whenever two of the variables are equal, since if $x_i = x_j$ then the i^{th} and j^{th} columns in the determinant are identical. Meanwhile, the right-hand side of (1) is, in some sense, the *simplest* polynomial in $x_1, \ldots x_n$ that evaluates to 0 whenever two of the variables are equal. (Well, "simplest" up to an overall sign. If we replace $x_j - x_i$ by $x_i - x_j$ on the right-hand side of (1), we get an equally '"simple" polynomial, but it differs from the original polynomial by a factor of $(-1)^{n(n-1/2}$, hence has the opposite sign when $n = 4m + 2$ or $4m + 3$ for some integer m.)

Example 4 [For Chapter 5 material.]

Let $A = \begin{pmatrix} 7 & 8 & 16 \\ 8 & -5 & 8 \\ 16 & 8 & 7 \end{pmatrix}$. Then the *charactistic polynomial* of A, with variable t, is

$$
\begin{aligned}
\det(A - tI) &= \begin{vmatrix} 7-t & 8 & 16 \\ 8 & -5-t & 8 \\ 16 & 8 & 7-t \end{vmatrix} \\
&\underset{R_3 \to R_3 - 2\,R_2}{=} \begin{vmatrix} 7-t & 8 & 16 \\ 8 & -5-t & 8 \\ 0 & 18+2t & -9-t \end{vmatrix} \\
&= (9+t)\begin{vmatrix} 7-t & 8 & 16 \\ 8 & -5-t & 8 \\ 0 & 2 & -1 \end{vmatrix} \quad \text{(factoring } 9+t \text{ out of row 3)} \\
&\underset{R_1 \to R_1 - 2R_2}{=} (9+t)\begin{vmatrix} -9-t & 18+2t & 0 \\ 8 & -5-t & 8 \\ 0 & 2 & -1 \end{vmatrix} \\
&= (9+t)^2\begin{vmatrix} -1 & 2 & 0 \\ 8 & -5-t & 8 \\ 0 & 2 & -1 \end{vmatrix} \quad \text{(factoring } 9+t \text{ out of row 1)} \\
&= (9+t)^2\left\{(-1)\begin{vmatrix} -5-t & 8 \\ 2 & -1 \end{vmatrix} - 2\begin{vmatrix} 8 & 8 \\ 0 & -1 \end{vmatrix} + 0 \cdot \text{something}\right\} \\
&= (9+t)^2\left\{-[(5+t)-16] - 2(-8)\right\} \\
&= (9+t)^2(27-t) \quad [\text{or}:\ -(t+9)^2(t-27)].
\end{aligned}
$$

This involved much less computation (and less chance for error) than expanding the original determinant along a row or column, obtaining the polynomial $-(t^3 - 9t^2 - 405t - 2187)$, and then trying to factor it. (However, the matrix A was fine-tuned to make row-reduction work out as simply as it did, with common factors involving t appearing in rows after some elementary operations were applied. Most matrices aren't nearly as cooperative! But, at the worst, we can always use elementary operations to put one or more 0's into a row or column, reducing the number of minors we need to compute if we expand along that row or column.)