Non-book problems for Assignment 11

Some of the facts you're asked to prove in these problems were proven in class—perhaps recently, perhaps several weeks ago. Do any such proofs over again (for practice and reinforcement); don't just say, "This is true because we proved it in class."

NB 11.1. Let V and W be finite-dimensional vector spaces, and let $T : V \to W$ be a linear map.

(a) Show that

$$\text{rank}(T) \le \min\{\dim(V), \dim(W)\}. \tag{1}$$

(b) Show that T is one-to-one if and only if $\text{rank}(T) = \dim(V)$.

(c) Show that T is onto if and only if $\text{rank}(T) = \dim(W)$.

Before starting on part (a), prove the following trivial lemma for yourself: for any real numbers x, y, z, the inequality "$x \le \min\{y, z\}$" is equivalent to "$x \le y$ and $x \le z$." After proving this lemma for yourself, don't bother citing it when you use it. Thus, for example, to show $\text{rank}(T) \le \min\{\dim(V), \dim(W)\}$ in the exercise above, you should show that $\text{rank}(T) \le \dim(V)$ and that $\text{rank}(T) \le \dim(W)$, and then say something like "Therefore $\text{rank}(T) \le \min\{\dim(V), \dim(W)\}$;" you should not break the argument down into cases according to which of $\dim(V)$ and $\dim(W)$ is the larger, and you don't need to cite the trivial lemma above.

Note: In the originally posted version of the NB 11 problems, the expressions "$\text{rank}(T)$," "$\dim(V)$," and "$\dim(W)$" in this paragraph were confusingly written as "$\text{rank}(S \circ T)$", "$\text{rank}(S)$", and "$\text{rank}(T)$" respectively. That was a cut-and-paste error. In last year's corresponding assignment, this paragraph was located after what is now NB 11.3(b), and before what is now NB 11.1.

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You should find the next two problems *quick* and *easy*. Everything that you're asked to check is an immediate consequence of *definitions*. If you're on top of the relevant definitions, you should be able to do most or all of the problem-parts in your head. The main purpose of these problems is to make you conscious of various facts that otherwise might not occur to you—however "obvious" or "trivial" they may seem once they're pointed out to you—and that can be very useful in proving less trivial statements.

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NB 11.2. Let V and W be vector spaces and let $T : V \to W$ be a linear transformation.

(a) Define $\overline{T} : V \to R(T)$ by $\overline{T}(v) = T(v)$ (for each $v \in V$). (Thus, the only difference between T and $\overline{T}$ is in their *codomains*.[1]) Check that the following are true:

(i) $\overline{T} : V \to R(T)$ is linear.

(ii) $R(\overline{T}) = R(T) = T(V)$.

(iii) $\overline{T}$ is onto. (This highlights the fact that T and $\overline{T}$ are not the same function, even though $\overline{T}(v) = T(v)$ for every $v \in V$.)

(iv) $N(\overline{T}) = N(T)$.

(b) Let $V_0 \subseteq V$ be a subspace. Recall that the *restriction of* T *to* V_0, denoted $T|_{V_0}$, is the function from V_0 to W defined by $T|_{V_0}(v) = T(v)$ (for each $v \in V_0$). (Thus, the only difference between T and $T|_{V_0}$ is in their *domains*.) Check that the following are true:

(i) $T|_{V_0} : V_0 \to W$ is linear.

(ii) $R(T|_{V_0}) = T(V_0)$.

(iii) $N(T|_{V_0}) = N(T) \cap V_0$.

NB 11.3. Let $V, W,$ and Z be vector spaces, let $T : V \to W$ and $S : W \to Z$ be linear transformations, and define $\overline{T}$ as in problem NB 11.1.

(a) Check that the following are true:

(i) $S \circ T = S|_{R(T)} \circ T = S|_{R(T)} \circ \overline{T}$.

(ii) $R(S \circ T) = S(R(T)) = R(S|_{R(T)})$

(Note: for general functions f and g, even if $\text{codomain}(f) \neq \text{domain}(g)$, the composition "$g \circ f$" is defined as long as the *range* of f is *contained in* the domain of g. In particular, the composition $S|_{R(T)} \circ T$ is defined even though $R(T)$ [the domain of $S|_{R(T)}$] may be not be all of W [the codomain of T].)

(b) Assume that V and W are finite-dimensional. Show that

$$\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}.$$

[This part (b) was not in the "NB 11" problems when Assignment 11 was posted; it was accidentally deleted when I edited the problem-list from my corresponding Fall 2023 assignment. Rather than inserting it into an "NB 12" problem-list, I'm inserting it into the NB 11 list because of its logical relation to earlier problems on the that list. The "quick and easy" problems NB 11.2 and the former version of NB 11.3—now part (a) above—are expanded versions of hints and

[1] Recall that a function f has three "parts": a specified domain, say A; a specified codomain, say B; and an assignment of an element $f(x) \in B$ to each element $x \in A$. Changing any of these "parts" creates a different function.

comments from last year's assignment. They provide insight into the current NB 11.3(b); that's the main reason I put them this problem list! But this year I also added NB 11.2 (a)(iii)–(iv) and 11.2(b)(iii), which are not relevant to NB 11.3(b).]

NB 11.4. Let V and W be finite-dimensional vector spaces and assume that $\dim(V) < \dim(W)$.

(a) Show that there exists a one-to-one linear transformation from V to W. (*Hint*: Theorem 2.6 in FIS Chapter 2.)

(b) Show that if a linear transformation $T : V \to W$ is one-to-one, then T has a *left inverse*, i.e. a linear transformation $S : W \to V$ such that $S \circ T = I_V$.

(c) Show that if a linear transformation $S : W \to V$ is onto, then S has a *right inverse*, i.e. a linear transformation $T : V \to W$ such that $S \circ T = I_V$. Show also that any such T is one-to-one. (*Note*: The S and T in this problem-part are *new*; they're not carried over from part (b). In fact, **the wording of part (b) tells you this. How?**)

(d) Give two proofs, as indicated below, that there do not exist any linear transformations $T : V \to W$ and $S : W \to V$ for which $T \circ S = I_W$:

- *Hint for Proof #1*: Use Problem NB 11.3.
- *Hint for Proof #2*: Show that for any sets X, Y and functions $f : X \to Y$ and $g : Y \to X$, if $g \circ f = I_X$ (the identity map of X), then f is one-to-one and g is onto. Then use what we proved several weeks ago about non-existence of certain linear maps from V to W, and certain linear maps from W to V, under the given assumption that $\dim(V) < \dim(W)$.

NB 11.5. Let V and W be finite-dimensional vector spaces of equal dimension n. Suppose that $T : V \to W$ and $S : W \to V$ are linear transformations for which $S \circ T = I_V$. Using the steps below, prove that $T \circ S = I_W$.

(a) Show that $\text{rank}(S) = \text{rank}(T) = n$. (*Hint*: Problem NB 11.1.) In particular, T is onto.

(b) Show that $T \circ S \circ T = T$. (Recall that composition of composable functions is associative: if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$. Hence both sides of the latter equation can unambiguously be denoted $h \circ g \circ f$.)

(c) Now use parts (a) and (b) to show that $T \circ S = I_W$.

Note: Problem NB 11.3(d) shows that **in order for "$S \circ T = I_V$" to imply "$T \circ S = I_W$"** (non-vacuously), **the assumption that** $\dim(V) = \dim(W)$ **is *crucial*!** (The "non-vacuously" means here that there exist *some* linear $T : V \to W$ and $S : W \to V$ for which $S \circ T = I_V$.)

NB 11.6. Let m and n be positive integers with $n < m$.

(a) Show that if $A \in M_{m\times n}(\mathbf{R})$ is such that the map $L_A : \mathbf{R}^n \to \mathbf{R}^m$ has rank n, then there exists a matrix $B \in M_{n\times m}(\mathbf{R})$ such that $BA = I_{n\times n}$.

(*Hint*: Use appropriate parts of Problem NB 11.4 and NB 11.5.)

(b) Show that there are no matrices $A \in M_{m\times n}(\mathbf{R}), B \in M_{n\times m}(\mathbf{R})$ such that $AB = I_{m\times m}$.

(Same hint.)

NB 11.7. Let $A, B \in M_{n\times n}(\mathbf{R})$ and let $I = I_{n\times n}$. Show that if $AB = I$, then $BA = I$. (*Hint*: Problem NB 11.4.)

NB 11.8. Let A and B be diagonal $n \times n$ matrices. Show that AB is also a diagonal matrix, and that if the diagonal entries of A are $\lambda_1, \ldots, \lambda_n$ and the diagonal entries of B are $\mu_1, \ldots, \mu_n$ (i.e. $\lambda_i = A_{ii}$ and $\mu_i = B_{ii},\ 1 \le i \le n$), then the diagonal entries of AB are simply the products $\lambda_1\mu_1, \ldots, \lambda_n\mu_n$.

NB 11.9. (a) Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Compute A^2, A^3, and A^4.

(b) In part (a), you should have found that A^4 is a very simple matrix—so simple that you can immediately tell what all higher powers of A would be. You should also have noticed a pattern in location of the nonzero above-diagonal entries in the sequence A, A^2, A^3. Does the value of A^4, or the pattern you noticed in the sequence A, A^2, A^3, depend at all on the values of the above-diagonal entries of A?

(c) For $n \times n$ matrices with $n \ge 2$, conjecture how your observations in part (b) would generalize.

(d) Try to prove the conjecture you made in part (c).

NB 11.10. Let $A, C \in M_{n\times n}(\mathbf{R})$ and assume that C is invertible. Show that for any integer $k \ge 1$,

$$(C^{-1}AC)^k = C^{-1}A^k\,C$$

and similarly

$$(CAC^{-1})^k = C\,A^k\,C^{-1}.$$

finite-dimensional. Let $T : V \to W$ and $S : W \to Z$ be linear transformations. Show that

$$\text{rank}(S \circ T) \le \min\{\text{rank}(S), \text{rank}(T)\}.$$

Before starting, prove the following trivial lemma for yourself: for any real numbers x, y, z, the inequality "$x \leq \min\{y, z\}$" is equivalent to "$x \leq y$ and $x \leq z$." After proving this lemma for yourself, don't bother citing it when you use it. Thus, for example, to show $\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}$ in the exercise above, you should show that $\text{rank}(S \circ T) \leq \text{rank}(S)$ and that $\text{rank}(S \circ T) \leq \text{rank}(T)$, and then say something like "Therefore $\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}$;" you should not break the argument down into cases according to which of $\text{rank}(S)$ and $\text{rank}(T)$ is the larger.

Something you may find helpful: for any vector spaces $\tilde{W}$ and $\tilde{Z}$, any subspace $X \subseteq \tilde{W}$, and any linear map $\tilde{S} : \tilde{W} \to \tilde{Z}$, the *restriction of* $\tilde{S}$ *to* X—i.e. the map $S|_X : X \to \tilde{Z}$ defined by $\tilde{S}|_X(x) = \tilde{S}(x)$ for all $x \in X$—is a linear map from X to $\tilde{Z}$ (why?).