An application of "eigenstuff": Computing the Fibonacci numbers

NB 14.1. Let $A \in M_{2\times 2}(\mathbf{R})$ and let $f_A(t) = \det(A - tI)$, the characteristic polynomial of A with the variable named t. Show that

$$f_A(t) = t^2 - \operatorname{tr}(A)t + \det(A). \tag{1}$$

(*Note*: For larger square matrices, the coefficients of the characteristic polynomial cannnot be expressed purely in terms of the trace and determinant of the matrix.)

NB 14.2 (Explicit formula for the Fibonacci numbers, via "eigenstuff") The *Fibonacci numbers* are the terms of the sequence

$$1, 1, 2, 3, 5, 8, 13, \ldots, \tag{2}$$

in which the first two terms are 1, and every term after that is the sum of the two previous terms. In this problem you will use matrix algebra (specifically, "eigenstuff") to compute an explicit formula for the Fibonacci numbers and some related sequences.

For $n \geq 1$ let f_n be the n^{th} term of the Fibonacci sequence (2). Thus $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. To simplify some formulas below, define $f_0 = 0$ (effectively, just inserting a 0 at the start of the sequence (2)), and observe that $f_2 = f_1 + f_0$, so that the recursive relation $f_{n+2} = f_{n+1} + f_n$ now holds for $n \geq 0$.

Define a sequence of vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ in $\mathbf{R}^2$ by

$$\mathbf{x}_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}. \tag{3}$$

(a) Show that this sequence of vectors $(\mathbf{x}_n)_{n=0}^{\infty}$ satisfies

$$\mathbf{x}_{n+1} = A\mathbf{x}_n \quad \text{for all } n \geq 0, \tag{4}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \tag{5}$$

Then use (4) to deduce that

$$\mathbf{x}_n = A^n\mathbf{x}_0 \quad \text{for all } n \geq 1. \tag{6}$$

(b) (i) Find the eigenvalues of A. You should find that these are two distinct, real numbers λ_1 and λ_2. Hence A is diagonalizable, so $\det(A) = \lambda_1\lambda_2$.

By problem NB 14.1, $\lambda_1\lambda_2 = \det(A) = -1$, so $|\lambda_1|\,|\lambda_2| = 1$. From your formula for the eigenvalues, you should *easily* (without a calculator) be able to see that one of the eigenvalues has absolute value greater than 1, so the other must have absolute value less than 1 (which can also be seen easily, but not quite *as* easily). Let λ_1 be the eigenvalue with $|\lambda_1| > 1$ and let λ_2 be the other eigenvalue. Below, let D be the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

(ii) Find eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ corresponding to λ_1, λ_2 respectively. Since $\lambda_1 \neq \lambda_2$, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly independent set in the two-dimensional vector space $\mathbf{R}^2$, hence is a basis of $\mathbf{R}^2$ (an A-eigenbasis).

(c) Express the vector $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. I.e. find $c_1, c_2 \in \mathbf{R}$ such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then show that

$$A^n\mathbf{x}_0 = c_1\,{\lambda_1}^n\,\mathbf{v}_1 + c_2\,{\lambda_2}^n\,\mathbf{v}_2\ , \tag{7}$$

and use equation (7) (together with equation (6) and the definition of $\mathbf{x}_n$) to write down an explicit formula for the n^{th} Fibonacci number f_n.

If you've done everything correctly, in your formula for f_n you'll see the irrational number $\sqrt{5}$ appearing in two fractions that are raised to higher and higher powers. Yet the Fibonacci numbers are integers! Remarkably, not only do all the $\sqrt{5}$'s cancel, allowing your formula to work out to a *rational* number for each n, the fractions "conspire" with each other to produce an *integer*.

(d) For each $n \geq 1$, compute D^n explicitly in terms of the eigenvalues of D. (Recall that for a diagonal matrix, the eigenvalues are precisely the diagonal entries.) Relate this to FIS exercise 5.1/16b.

(e) Use the information found in part (b) to construct an invertible matrix $C \in M_{2\times 2}(\mathbf{R})$ such that $D = C^{-1}AC$.

Since $D = C^{-1}AC$, we also have $A = CDC^{-1}$ (why?). From non-book problem NB 11.10, we then have

$$A^n = CD^nC^{-1} \quad \text{for any } n \geq 1. \tag{8}$$

Since $\mathbf{x}_n = A^n\mathbf{x}_0$, we now have two more ways of computing $\mathbf{x}_n$:

(i) Compute A^n explicitly, and then multiply $\mathbf{x}_0$ by the result.

(ii) Use the associativity of matrix multiplication to compute $A^n\mathbf{x}_0 = CD^nC^{-1}\mathbf{x}_0$ without ever computing A^n itself, by doing the matrix computation in "right-to-left" order (as indicated by the parentheses below):

$$\underbrace{C\big(\overbrace{D^n(\underbrace{C^{-1}\mathbf{x}_0}_{\text{compute first}})}^{\text{compute second}}\big)}_{\text{compute third}}. \tag{9}$$

Note that methods (i), (ii), and the method in part (c) should all give the same answer! After you've done the computation all three ways, compare the methods and see how and where the same information is packaged differently.