Non-book problems for Assignment 6

An infinite set A is called *countably infinite* (or simply *countable*, if the "infinite" is understood from context) if there is a one-to-one correspondence between A and the $\mathbf{N}$ (the set of positive integers), i.e. a bijection from $\mathbf{N}$ to A or, equivalently, a bijection from A to $\mathbf{N}$. This is equivalent to the statement that the elements of A can be *enumerated*; i.e. that A can be written as $\{a_1, a_2, a_3, \dots\}$.

Fact: every infinite subset of a countably infinite set is countably infinite. (This is not hard to prove, but just assume it for the sake of this problem-set.)

Not all infinite sets are countable. Those that aren't are called *uncountable.* Uncountable sets are "more infinite" than countably infinite sets (in a sense that can be made precise, which I'm not doing here). It can be shown that the real numbers are uncountable (i.e. that the set $\mathbf{R}$ is uncountable).

Reminder (unrelated to the preceding): Earlier this semester, we showed that for any nonempty set S, the set $\mathcal{F}(S, \mathbf{R})$ of ALL functions from S to $\mathbf{R}$, with addition and scalar multiplication operations defined "pointwise" (meaning the way we did this in class), is a vector space.

NB 6.1. We have seen that the countable set $\beta = \{1, x, x^2, \dots\}$ is a basis of $P(\mathbf{R})$. Consider the sets $\beta_1 := \beta \setminus \{1\} = \{x, x^2, x^3, \dots\}$ and $\beta_{\text{ev}} := \{(1, x^2, x^4, x^6, \dots\}$.

(a) Show that each of the sets β_1 and β_{ev} is an infinite, linearly independent set that does not span $P(\mathbf{R})$. (You may take for granted that these sets are infinite; I've stated that property just for emphasis.)

(b) Describe $\text{span}(\beta_1)$ and $\text{span}(\beta_{\text{ev}})$ without using any linear-algebraic terms. (In each case, the span consists of functions with certain easily-identified properties that involve no linear-algebraic terminology.)

NB 6.2. In FIS exercise 1.6/ 21, half of what you showed is that if V is an infinite-dimensional vector space, then V contains an infinite linearly independent subset L. Show that if V is an infinite-dimensional vector space then V has a countably infinite, linearly independent set that does *not* span V. (This is true whether or not V has a countably infinite, linearly independent set that *does* span V, i.e. a countably infinite basis.) *Hint*: problem NB6.1(a).

NB 6.3. Consider the vector space $\mathcal{F}(\mathbf{R}, \mathbf{R})$. For each $c \in \mathbf{R}$, define $f_c \in \mathcal{F}(\mathbf{R}, \mathbf{R})$ by

$$f_c(x) = \begin{cases} 1 & \text{if } x = c, \\ 0 & \text{if } x \neq c. \end{cases}$$

Let $B = \{f_c : c \in \mathbf{R}\}$. Note that B is an uncountably infinite subset of $\mathcal{F}(\mathbf{R}, \mathbf{R})$.

(a) Show that the set B is linearly independent.

(b) Show that B does *not* span $\mathcal{F}(\mathbf{R}, \mathbf{R})$, and hence is not a basis of this vector space.

(c) In clear, plain English, describe which functions from $\mathbf{R}$ to $\mathbf{R}$ lie in span(B). Here, "[i]n clear, plain English" means that you should not use any linear-algebra terminology (other than "span(B)" itself), or invent your own terminology, or use words or phrasing that might require mind-reading on the part of the reader.

Remark. The fact that $\mathcal{F}(\mathbf{R}, \mathbf{R})$ has an *uncountable* linearly independent set makes it plausible that $\mathcal{F}(\mathbf{R}, \mathbf{R})$ does not have a countable basis. This plausible fact is actually true, but a proof is beyond the level of this course. Also beyond the level of this course is the fact that $\mathcal{F}(\mathbf{R}, \mathbf{R})$ *has a basis at all.* The failure of B to be a basis of $\mathcal{F}(\mathbf{R}, \mathbf{R})$ makes it hard to imagine (at least for me!) what subset(s) of $\mathcal{F}(\mathbf{R}, \mathbf{R})$ *could* be a basis. But it can be shown that *every* vector space has a basis. (A proof of this, too, is above the level of this course. But for the interested student who's acquainted with the Axiom of Choice, there's a proof in Section 1.7.)

NB 6.4. (Partial generalization of problem NB 6.3.) Let S be a nonempty set, and consider the vector space $\mathcal{F}(S, \mathbf{R})$. For each $a \in S$, define $f_a : S \to \mathbf{R}$ (equivalently, $f_a \in \mathcal{F}(S, \mathbf{R})$) by

$$f_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \neq a. \end{cases}$$

Let $B_S = \{f_a : a \in S\}$.

(a) Show that the set B_S is linearly independent.

(b) In clear, plain English, describe which functions from S to $\mathbf{R}$ lie in span(B_S).

(c) Under what condition(s) on S does B_S span $\mathcal{F}(S, \mathbf{R})$?