Non-book problems for Assignment 8

Before starting, review the definition of "external" direct sum in the Direct Sums handout.

NB 8.1. Let W_1 and W_2 be finite-dimensional subspaces of a vector space V. In an earlier homework exercise (FIS 1.6/ 29(a)) you showed that $W_1 + W_2$ is finite-dimensional and that

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2). \tag{1}$$

In this problem, you will give a different proof of formula (1).

Since W_1 and W_2 are vector spaces, the "external" direct sum $W_1 \oplus_e W_2$ is defined. (Note that $W_1 \oplus_e W_2$ is not the same vector space, or even the same *set*, as the subspace $W_1 + W_2 \subseteq V$.) Let

$$K = \{(v, -v) : v \in W_1 \cap W_2\} \subseteq W_1 \oplus_e W_2.$$

(a) Show that K is a subspace of $W_1 \oplus_e W_2$.

(b) Show that K is finite-dimensional and that $\dim(K) = \dim(W_1 \cap W_2)$.

(c) Define $T : W_1 \oplus_e W_2 \to W_1 + W_2$ by $T(w_1, w_2) = w_1 + w_2$.
    (i) Show that T is linear.
    (ii) Show that $R(T) = W_1 + W_2$.
    (iii) Show that $N(T) = K$.

(d) Recall from FIS exercise 1.6/ 25 (in Assignment 4) that $W_1 \oplus_e W_2$ is also finite-dimensional and that

$$\dim(W_1 \oplus_e W_2) = \dim(W_1) + \dim(W_2). \tag{2}$$

Note also that $W_1 \cap W_2$ is finite-dimensional, since it's a subspace of W_1. (We could equally well have said "since it's is a subspace of W_2.")

Using equation (2), earlier parts of this problem, and the Rank-Plus-Nullity Theorem, show that $W_1 + W_2$ is finite-dimensional, and prove the dimensional relation (1).

NB 8.2. In the previous problem, we assumed at the outset that both W_1 and W_2 were finite-dimensional. This assumption was more than was needed for several parts of the problem.

(a) For which parts of problem NB 8.1 could we have eliminated *all* assumptions of finite-dimensionality?

(b) For which parts of problem NB 8.1 would it have been enough to assume that *at least one* of W_1 and W_2 is finite-dimensional?

(c) Which parts of problem NB 8.1 actually used the assumption that *both* W_1 and W_2 are finite-dimensional?

NB 8.3. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. (Note that this is not the same hypothesis as in NB 8.1.) Show that if any two of the conditions (i), (ii), and (iii) are satisfied, then so is the third:

 (i) $W_1 \cap W_2 = \{0_V\}$.

 (ii) $W_1 + W_2 = V$.

 (iii) $\dim(W_1) + \dim(W_2) = \dim(V)$.

Hence if any two of (i)–(iii) are satisfied, then V is the "internal" direct sum of W_1 and W_2. (This problem is an extension of FIS 1.6/29(b), in which you proved "(ii) + (iii) $\implies$ (i).")