Non-book problems for Assignment 11

NB 11.1. Let A and B be diagonal $n \times n$ matrices. Show that AB is also a diagonal matrix, and that if the diagonal entries of A are $\lambda_1, \ldots, \lambda_n$ and the diagonal entries of B are $\mu_1, \ldots, \mu_n$ (i.e. $\lambda_i = A_{ii}$ and $\mu_i = B_{ii},\ 1 \leq i \leq n$), then the diagonal entries of AB are simply the products $\lambda_1\mu_1, \ldots, \lambda_n\mu_n$.

NB 11.2. (a) Let $A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Compute A^2, A^3, and A^4.

(b) In part (a), you should have found that A^4 is a very simple matrix—so simple that you can immediately tell what all higher powers of A would be. You should also have noticed a pattern in location of the nonzero above-diagonal entries in the sequence A, A^2, A^3. Does the value of A^4, or the pattern you noticed in the sequence A, A^2, A^3, depend at all on the values of the above-diagonal entries of A?

(c) For $n \times n$ matrices with $n \geq 2$, conjecture how your observations in part (b) would generalize.

(d) Try to prove the conjecture you made in part (c).

NB 11.3. Let $A, C \in M_{n\times n}(\mathbf{R})$ and assume that C is invertible. Show that for any integer $k \geq 1$,

$$(C^{-1}AC)^k = C^{-1}A^k\, C$$

and similarly

$$(CAC^{-1})^k = C\, A^k\, C^{-1}.$$

NB 11.4. *Matrix model for the complex number system.* If you need to review complex numbers before doing this problem, see Appendix D in FIS.

(a) Check that $\mathbf{C}$, the space of complex numbers, is a real vector space for which $\{1, i\}$ is a basis. (Hence the dimension of this vector space is two.)

(b) In $M_{2\times 2}(\mathbf{R})$, let

$$I \;=\; I_{2\times 2} \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and let $H = \text{span}\{I, J\} \subseteq M_{2\times 2}(\mathbf{R})$. Clearly $\{I, J\}$ is a linearly independent set, so H is a two-dimensional (real) vector space for which $\{I, J\}$is a basis.

Check that H is the space of real matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

(c) Compute J^2 and express your answer in terms of I.

(d) Show that H is closed under multiplication (of 2×2 matrices). I.e. show that if $Z, W \in H$, then $ZW \in H$.

(e) Define $\phi : H \to \mathbf{C}$ to be the linear map for which $\phi(I) = 1$ and $\phi(J) = i$. (Thus $\phi(aI + bJ) = a + bi$ for all $a, b \in \mathbf{R}$.) Show ϕ is an isomorphism (of vector spaces) and that, in addition,

$$\phi(ZW) \quad = \quad \phi(Z)\phi(W) \quad \text{for all } Z, W \in H. \tag{1}$$

Thus $\phi : H \to \mathbf{C}$ is a bijective map that carries matrix addition (of matrices in H) to addition of complex numbers, and carries matrix multiplication (of matrices in H) to multipliication of complex numbers.

(f) In the usual introduction to complex numbers, we define the multiplication operation by declaring the product $(a + bi)(c + di)$ to be $ac - bd + (ad + bc)i$ (where $a, b, c, d \in \mathbf{R}$). The question then arises: is this operation associative? To verify that it *is* associative, we then take three arbitrary complex numbers, say $z_1 = a + bi$, $z_2 = c + di$, $z_3 = e + fi$ (where $a, b, c, d, e, f \in \mathbf{R}$), compute $(z_1 z_2)z_3$ and $z_1(z_2 z_3)$, and check that the results are equal.

Give a different proof of the associativity of complex multiplication, using the map ϕ and the associativity of matrix multiplication.