

Non-book problems for Assignment 14

NB 14.1. Let $A \in M_{2 \times 2}(\mathbf{R})$ and let $f_A(t) = \det(A - tI)$, the characteristic polynomial of A with the variable named t . Show that

$$f_A(t) = t^2 - \operatorname{tr}(A)t + \det(A). \quad (1)$$

(Note: The simplicity of equation (1) is special to 2×2 matrices. In the next exercise, you will figure out what (1) generalizes to when $n > 2$.)

NB 14.2. Let k, n be integers with $1 \leq k \leq n$. The k^{th} elementary symmetric function σ_k of variables x_1, x_2, \dots, x_n is defined by

$$\sigma_k(x_1, \dots, x_n) = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}.$$

(The sum is over all ordered k -tuples of integers (i_1, i_2, \dots, i_k) satisfying $1 \leq i_1 < i_2 < \dots < i_k \leq n$.) For example, with $n = 3$,

$$\begin{aligned} \sigma_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3, \\ \sigma_2(x_1, x_2, x_3) &= x_1 x_2 + x_1 x_3 + x_2 x_3, \quad \text{and} \\ \sigma_3(x_1, x_2, x_3) &= x_1 x_2 x_3. \end{aligned}$$

(The Greek letter σ is a lower-case sigma.)

(a) Let $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbf{R}$ and let D be the $n \times n$ diagonal matrix with $D_{ii} = \lambda_i, 1 \leq i \leq n$. Show that f_D , the characteristic polynomial of D , satisfies

$$\begin{aligned} (-1)^n f_D(t) &= (t - \lambda_1)(t - \lambda_2) \dots (t - \lambda_n) \\ &= t^n - \sigma_1(\lambda_1, \dots, \lambda_n) t^{n-1} + \sigma_2(\lambda_1, \dots, \lambda_n) t^{n-2} \\ &\quad - \dots + (-1)^n \sigma_n(\lambda_1, \lambda_2, \dots, \lambda_n). \end{aligned} \quad (2)$$

(b) Using part (a), show that if A is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (not necessarily distinct; an eigenvalue of algebraic multiplicity m appears m times in this list) then the characteristic polynomial $f_A(t)$ is also given by the right-hand side of equation (2). In particular, the coefficient of t^{n-1} in $f_A(t)$ is always $(-1)^{n-1} \operatorname{tr}(A)$, and furthermore

$$\begin{aligned} \det(A) &= \det(A - tI)|_{t=0} = f_A(0) \\ &= \text{constant term in } f_A(t) \\ &= \sigma_n(\lambda_1, \lambda_2, \dots, \lambda_n) \\ &= \lambda_1 \lambda_2 \dots \lambda_n. \end{aligned}$$

Thus the determinant of A is the product of its eigenvalues (counted with multiplicity):

$$\det(A) = \lambda_1 \lambda_2 \dots \lambda_n. \quad (3)$$

NB 14.3. The *Fibonacci numbers* are the terms of the sequence

$$1, 1, 2, 3, 5, 8, 13, \dots, \quad (4)$$

in which the first two terms are 1, and every term after that is the sum of the two previous terms. In this problem you will use matrix algebra (specifically, “eigenstuff”) to compute an explicit formula for the Fibonacci numbers and some similarly defined sequences.

For $n \geq 1$ let f_n be the n^{th} term of the Fibonacci sequence (4). Thus $f_1 = f_2 = 1$ and $f_{n+2} = f_{n+1} + f_n$ for $n \geq 1$. To simplify some formulas below, define $f_0 = 0$ (effectively, just inserting a 0 at the start of the sequence (4)), and observe that $f_2 = f_1 + f_0$, so that the recursive relation $f_{n+2} = f_{n+1} + f_n$ now holds for $n \geq 0$.

Define a sequence of vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \dots$ in \mathbf{R}^2 by

$$\mathbf{x}_n = \begin{pmatrix} f_n \\ f_{n+1} \end{pmatrix}. \quad (5)$$

(a) Show that this sequence of vectors $(\mathbf{x}_n)_{n=0}^\infty$ satisfies

$$\mathbf{x}_{n+1} = A\mathbf{x}_n \quad \text{for all } n \geq 0, \quad (6)$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}. \quad (7)$$

Then use (6) to deduce that

$$\mathbf{x}_n = A^n \mathbf{x}_0, \quad \text{for all } n \geq 1. \quad (8)$$

(b) (i) Find the eigenvalues of A . You should find that these are two distinct, real numbers λ_1 and λ_2 .

By problem NB 14.1, $\lambda_1 \lambda_2 = \det(A) = -1$, so $|\lambda_1| |\lambda_2| = 1$. From your formula for the eigenvalues, you should *easily* (without a calculator) be able to see that one of the eigenvalues has absolute value greater than 1, so the other must have absolute value less than 1 (which can also be seen easily, but slightly less easily). Let λ_1 be the eigenvalue with $|\lambda_1| > 1$ and let λ_2 be the other eigenvalue. Below, let D be the diagonal matrix $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

(ii) Find eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ corresponding to λ_1, λ_2 respectively. Since $\lambda_1 \neq \lambda_2$, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent set in the two-dimensional vector space \mathbf{R}^2 , hence is a basis of \mathbf{R}^2 (an A -eigenbasis).

(c) Express the vector $\mathbf{x}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in terms of \mathbf{v}_1 and \mathbf{v}_2 . I.e. find $c_1, c_2 \in \mathbf{R}$ such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then show that

$$A^n \mathbf{x}_0 = c_1 \lambda_1^n \mathbf{v}_1 + c_2 \lambda_2^n \mathbf{v}_2, \quad (9)$$

and use equation (9) (together with equation (8) and the definition of \mathbf{x}_n) to write down an explicit formula for the n^{th} Fibonacci number f_n .

If you've done everything correctly, in your formula for f_n you'll see the irrational number $\sqrt{5}$ appearing in two fractions that are raised to higher and higher powers. Yet the Fibonacci numbers are integers! Remarkably, not only do all the $\sqrt{5}$'s cancel, allowing your formula to work out to a *rational* number for each n , the fractions "conspire" with each other to produce an *integer*.

(d) For each $n \geq 1$, compute D^n explicitly in terms of the eigenvalues of D . (Recall that for a diagonal matrix, the eigenvalues are precisely the diagonal entries.) Relate this to FIS exercise 5.1/16b.

(e) Use the information found in part (b) to construct an invertible matrix $C \in M_{2 \times 2}(\mathbf{R})$ such that $D = C^{-1}AC$.

Since $D = C^{-1}AC$, we also have $A = CDC^{-1}$ (why?). From non-book problem NB 11.3, we then have

$$A^n = CD^nC^{-1} \quad \text{for any } n \geq 1. \quad (10)$$

Since $\mathbf{x}_n = A^n \mathbf{x}_0$, we now have two more ways of computing \mathbf{x}_n :

(i) Compute A^n explicitly, and then multiply \mathbf{x}_0 by the result.

(ii) Use the associativity of matrix multiplication to compute $A^n \mathbf{x}_0 = CD^nC^{-1} \mathbf{x}_0$ without ever computing A^n itself, by doing the matrix computation in "right-to-left" order (as indicated by the parentheses below):

$$\underbrace{C \left(\overbrace{D^n \left(\underbrace{C^{-1} \mathbf{x}_0}_{\text{compute first}} \right)}^{\text{compute second}} \right)}_{\text{compute third}}. \quad (11)$$

Note that methods (i), (ii), and the method in part (c) should all give the same answer! After you've done the computation all three ways, compare the methods and see how and where the same information is packaged differently.

(f) (i) Compute $\lim_{n \rightarrow \infty} \frac{f_{n+1}}{f_n}$ and relate the answer to eigenvalue(s) of A (equivalently, eigenvalues of D). Then look back at equation (9) and see how you could have predicted the value of this limit.

(ii) Consider a modified Fibonacci sequence $(a_n)_{n=0}^{\infty}$ in which the first two terms are replaced by arbitrary numbers a_0 and a_1 , not both 0, but the recursive rule $a_n = a_{n-1} + a_{n-2}$ is still used for $n \geq 2$. Using (9), show that this limit is “almost” independent of the initial terms a_0 and a_1 , in the following sense: that the limit always exists, and is always either λ_1 or λ_2 , but is λ_1 unless the vector $\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$ lies in the λ_2 -eigenspace of A .

(g) **This part is not homework.**

Summer fun. Let $a, b \in \mathbf{R}$ and consider the sequence $(y_n)_{n=0}^{\infty}$ defined recursively by choosing y_0 and y_1 arbitrarily and setting

$$y_n = ay_{n-1} + by_{n-2} \quad \text{for } n \geq 2.$$

(i) Assume that $a^2 + 4b > 0$. Use methods from earlier parts of this problem to find an explicit formula for y_n ($n \geq 2$) in terms of y_0 and y_1 .

(ii) See if you can figure out the analogous formulas in the cases $a^2 + 4b < 0$ and $a^2 + 4b = 0$. (We did not cover all the material that would help you do this!)