Non-book problems for Assignment 15

In the problems below, $(V, \langle\ ,\ \rangle)$ is a finite-dimensional inner-product space of positive dimension n.

NB 15.1. (a) Let H be a subspace of V. In class, we defined a map $\mathrm{proj}_H : V \to H$ called the *orthogonal projection from V to H.* We also showed that $V = H \oplus H^\perp$. You saw maps that were called *projections* earlier this semester, in the exercises for FIS Section 2.1: given any subspaces W_1, W_2 of V for which $V = W_1 \oplus W_2$, a map that the book called *the projection (of V) on W_1 along W_2* was defined. Show that, in this terminology, the map proj_H is precisely the projection of V on H along $H^\perp$.

(b) Show that, for all $v \in V$, the relation

$$\mathrm{proj}_H(v) + \mathrm{proj}_{H^\perp}(v) = v. \tag{1}$$

(Equivalently, in the vector space $\mathcal{L}(V, V)$, we have

$$\mathrm{proj}_H + \mathrm{proj}_{H^\perp} = I_V\ . \tag{2}$$

(c) Do the analogs of equation (1) and (2) hold for *any* direct-sum decomposition of an arbitrary vector space V' as $W_1 \oplus W_2$, where W_1, W_2 are complementary subspace of V'), or is it special to inner-product spaces and orthgonal complements?

NB 15.2 Consider the case in which $V = \mathbf{R}^n$, where $n \geq 2$, and $\langle\ ,\ \rangle$ is the standard inner product on $\mathbf{R}^n$. (For notational ease, in this problem I'll write elements of $\mathbf{R}^n$ just as ordered n-tuples rather than as column vectors.) Let $\{\mathbf{e}_i\}_{i=1}^n$ be the standard ordered basis of $\mathbf{R}^n$.

For $k \in \{1.2, \ldots, n-1\}$ let

$$\begin{aligned} H_k &= \{(a_1, \ldots, a_n) \in \mathbf{R}^n : a_i = 0 \text{ for all } i > k\} \\ &= \{(a_1, \ldots, a_k, 0, 0, \ldots, 0) : a_1, \ldots, a_k \in \mathbf{R}\} \\ &= \mathrm{span}\{\mathbf{e}_1, \ldots, \mathbf{e}_k\}. \end{aligned}$$

(a) Explicitly, what is the subspace $(H_k)^\perp$?

(b) Let $\mathbf{a} = (a_1, \ldots, a_n)$ and let $k \in \{1, 2, \ldots, n-1\}$. Compute $\mathrm{proj}_{H_k}(\mathbf{a})$ and $\mathrm{proj}_{(H_k)^\perp}(\mathbf{a})$ in terms of $a_1, \ldots a_n$. (Once you find the first of these, you should be able to write down the the second immediately, without any real computations, from earlier problem(s) on this page.)

NB 15.3. Let $\beta = \{u_1, \ldots, u_n\}$ be an orthonormal basis of V, and let $v, w \in V$. Let $\mathbf{a} = [v]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ and $\mathbf{b} = [w]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$. (Thus $v = \sum_i a_i u_i$ and $w = \sum_i b_i u_i$.)

(a) Show that

$$\langle v, w \rangle = \sum_{i=1}^{n} a_i b_i = [v]_\beta \cdot [w]_\beta. \tag{3}$$

In other words, in an orthonormal basis of $(V, \langle\ ,\ \rangle)$, the inner product "looks just like" dot-product. (**This is NOT true in an *arbitrary* basis!!**)

(b) From equation (3), deduce that if $a_1, \ldots, a_n$ are the coordinates of a vector $v \in V$ with respect to an orthonormal basis,

$$\|v\|^2 = {a_1}^2 + {a_2}^2 + \cdots + {a_n}^2\ . \tag{4}$$

(**Again, this is NOT true for an *arbitrary* basis!!**)