

## Non-book problems for Assignment 5

An infinite set  $A$  is called *countably infinite* (or simply *countable*, if the "infinite" is understood from context) if there is a one-to-one correspondence between  $A$  and the natural numbers.  $\mathbf{N}$  (i.e. a bijection from  $\mathbf{N}$  to  $A$  or, equivalently, a bijection from  $A$  to  $\mathbf{N}$ ). This is equivalent to the statement that the elements of  $A$  can be *enumerated*; i.e. that  $A$  can be written as  $\{a_1, a_2, a_3, \dots\}$ .

*Fact:* every infinite subset of a countably infinite set is countably infinite. (This is not hard to prove, but just assume for the sake of this problem-set.)

Not all infinite sets are countable. Those that aren't are called *uncountable*. Uncountable sets are "more infinite" than countably infinite sets (in a sense that can be made precise, which I'm not doing here). It can be shown that the real numbers are uncountable (i.e. that the set  $\mathbf{R}$  is uncountable).

**Reminder** (unrelated to the preceding): Earlier this semester, we showed that for any nonempty set  $S$ , the set  $\mathcal{F}(S, \mathbf{R})$  of ALL functions from  $S$  to  $\mathbf{R}$ , with addition and scalar multiplication operations defined "pointwise" (meaning the way we did this in class), is a vector space.

**NB 5.1.** We have seen that the countable set  $\beta = \{1, x, x^2, \dots\}$  is a basis of  $P(\mathbf{R})$ . Consider the sets  $\beta_1 := \beta \setminus \{1\} = \{x, x^2, x^3, \dots\}$  and  $\beta_{\text{ev}} := \{(1, x^2, x^4, x^6, \dots)\}$ .

- (a) Show that each of the sets  $\beta_1$  and  $\beta_{\text{ev}}$  is an infinite, linearly independent set that does not span  $P(\mathbf{R})$ . (You may take for granted that these sets are infinite; I've stated that property just for emphasis.)
- (b) Describe  $\text{span}(\beta_1)$  and  $\text{span}(\beta_{\text{ev}})$  without using any linear-algebraic terms. (In each case, the span consists of functions with certain easily-identified properties that involve no linear-algebraic terminology.)

**NB 5.2.** In FIS exercise 1.6/ 21, half of what you showed is that if  $V$  is an infinite-dimensional vector space, then  $V$  contains an infinite linearly independent subset  $L$ . Show that if  $V$  is an infinite-dimensional vector space then  $V$  has a countably infinite, linearly independent set that does *not* span  $V$ . (This is true whether or not  $V$  has a countably infinite that *does* span  $V$ , i.e. a basis.) *Hint:* problem B5.1(a).

**NB 5.3.** Consider the vector space  $\mathcal{F}(\mathbf{R}, \mathbf{R})$ . For each  $c \in \mathbf{R}$ , define  $f_c \in \mathcal{F}(\mathbf{R}, \mathbf{R})$  by

$$f_c(x) = \begin{cases} 1 & \text{if } x = c, \\ 0 & \text{if } x \neq c. \end{cases}$$

Let  $B = \{f_c : c \in \mathbf{R}\}$ . Note that  $B$  is an uncountably infinite subset of  $\mathcal{F}(\mathbf{R}, \mathbf{R})$ .

- (a) Show that the set  $B$  is linearly independent.
- (b) Show that  $B$  does *not* span  $\mathcal{F}(\mathbf{R}, \mathbf{R})$ , and hence is not a basis of this vector space.
- (c) In clear, plain English, describe which functions from  $\mathbf{R}$  to  $\mathbf{R}$  lie in  $\text{span}(B)$ . Here, “[i]n clear, plain English” means that you should not use any linear-algebra terminology (other than “ $\text{span}(B)$ ” itself), or invent your own terminology, or use words or phrasing that might require mind-reading on the part of the reader.

**Remark.** The fact that  $\mathcal{F}(\mathbf{R}, \mathbf{R})$  has an *uncountable* linearly independent set makes it plausible that  $\mathcal{F}(\mathbf{R}, \mathbf{R})$  does not have a countable basis. This plausible fact is actually true, but a proof is beyond the level of this course. Also beyond the level of this course is the fact that  $\mathcal{F}(\mathbf{R}, \mathbf{R})$  *has a basis at all*. The failure of  $B$  to be a basis of  $\mathcal{F}(\mathbf{R}, \mathbf{R})$  makes it hard to imagine (at least for me!) what subset(s) of  $\mathcal{F}(\mathbf{R}, \mathbf{R})$  *could* be a basis. But it can be shown that *every* vector space has a basis. (A proof of this, too, is above the level of this course. But for the interested student who’s acquainted with the Axiom of Choice, there’s a proof in Section 1.7.)

**NB 5.4.** (Partial generalization of problem NB 5.3.) Let  $S$  be a nonempty set, and consider the vector space  $\mathcal{F}(S, \mathbf{R})$ . For each  $a \in S$ , define  $f_a : S \rightarrow \mathbf{R}$  (equivalently,  $f_a \in \mathcal{F}(S, \mathbf{R})$ ) by

$$f_a(x) = \begin{cases} 1 & \text{if } x = a, \\ 0 & \text{if } x \neq a. \end{cases}$$

Let  $B_S = \{f_a : a \in S\}$ .

- (a) Show that the set  $B_S$  is linearly independent.
- (b) In clear, plain English, describe which functions from  $S$  to  $\mathbf{R}$  lie in  $\text{span}(B_S)$ .
- (c) Under what condition(s) on  $S$  does  $B_S$  span  $\mathcal{F}(S, \mathbf{R})$ ?