Non-book problems for Assignment 9

NB 9.1. Let V, W, Z be vector spaces, with V and W finite-dimensional. Let $T : V \to W$ and $S : W \to Z$ be linear transformations. Show that

$$\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}.$$

Before starting, prove the following trivial lemma for yourself: for any real numbers x, y, z, the inequality "$x \leq \min\{y, z\}$" is equivalent to "$x \leq y$ and $x \leq z$." After proving this lemma for yourself, don't bother citing it when you use it. Thus, for example, to show $\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}$ in the exercise above, you should show that $\text{rank}(S \circ T) \leq \text{rank}(S)$ and that $\text{rank}(S \circ T) \leq \text{rank}(T)$, and then say something like "Therefore $\text{rank}(S \circ T) \leq \min\{\text{rank}(S), \text{rank}(T)\}$;" you should not break the argument down into cases according to which of $\text{rank}(S)$ and $\text{rank}(T)$ is the larger.

NB 9.2. Let $m, n, p \in \mathbf{N}$.

(a) Let $A \in M_{m \times n}(\mathbf{R})$ and $B \in M_{n \times m}(\mathbf{R})$. Then both the products AB and BA are defined, and both are square matrices (the first is $m \times m$; the second is $n \times n$), so the trace of each is defined. Show that

$$\text{tr}(AB) = \text{tr}(BA), \tag{1}$$

whether or not $m = n$. (This generalizes the first part of FIS exercise 2.3/13, in which you showed that equation (1) holds if both A and B are $n \times n$.)

(b) Let $A \in M_{m \times n}(\mathbf{R}), B \in M_{n \times m}(\mathbf{R})$, and $C \in M_{n \times p}(\mathbf{R})$. Check that each of the products ABC, BCA, and CAB is defined and is a square matrix, and show that

$$\text{tr}(ABC) = \text{tr}(BCA) = \text{tr}(CAB). \tag{2}$$

(Use the *result of* part (a) to do this very quickly; don't give a lengthier version of the *argument* you used for part (a).)

Observe that the permutations of "A, B, C" appearing in equation (2) are only the *cyclic permutations*, not *all* permutations. Equation (2) (as well as its generalization in part (c)), is often called the "cyclic property of the trace."

(c) Formulate and prove a generalization of part (b) for arbitrarily many appropriately-sized matrices.