Non-book problem for Assignment 14

NB 14.1. Let k, n be integers with $1 \le k \le n$. The k^{th} *elementary symmetric function* σ_k of variables $x_1, x_2, \ldots, x_n$ is defined by

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

(The sum is over all ordered k-tuples of integers $(i_1, i_2, \ldots, i_n)$ satisfying $1 \le i_1 < i_2 < \cdots < i_k \le n$.) For example, with $n = 3$,

$$\begin{aligned} \sigma_1(x_1, x_2, x_3) &= x_1 + x_2 + x_3 \ , \\ \sigma_2(x_1, x_2, x_3) &= x_1\, x_2 + x_1\, x_3 + x_2\, x_3 \ , \quad \text{and} \\ \sigma_3(x_1, x_2, x_3) &= x_1\, x_2\, x_3 \ . \end{aligned}$$

(The Greek letter σ is a lower-case sigma.) These functions are called *symmetric* because interchanging any two of the variables—indeed, permuting the variables any way at all—does not change the value of the function. E.g,, as you may check,

$$\sigma_2(x_1, x_2, x_3) = \sigma_2(x_1, x_3, x_2) = \sigma_2(x_3, x_2, x_1) = \sigma_2(x_2, x_1, x_3) = \sigma_2(x_2, x_3, x_1) = \sigma_2(x_3, x_1, x_2).$$

Observe that for any n, the first and last elementary symmetric functions are just the sum and product of the variables:

$$\begin{aligned} \sigma_1(x_1, x_2, \ldots, x_n) &= x_1 + x_2 + \cdots + x_n \\ \text{and} \qquad \sigma_n(x_1, x_2, \ldots, x_n) &= x_1\, x_2 \cdots x_n \ . \end{aligned}$$

(The σ_k wih $1 < k < n$ are less simple.)

(a) Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{R}$ and let D be the $n \times n$ diagonal matrix with $D_{ii} = \lambda_i, 1 \le i \le n$. Show that f_D, the characteristic polynomial of D, satisfies

$$\begin{aligned} (-1)^n f_D(t) \ = \ \det(tI - D) \ &= \ (t - \lambda_1)(t - \lambda_2) \ldots (t - \lambda_n) \\ &= \ t^n \ - \ \sigma_1(\lambda_1, \ldots, \lambda_n)\, t^{n-1} \ + \ \sigma_2(\lambda_1, \ldots, \lambda_n)\, t^{n-2} \\ &\quad - \ \ldots \ + (-1)^n \sigma_n(\lambda_1, \lambda_2, \ldots, \lambda_n). \end{aligned} \tag{1}$$

(b) Using part (a), show that if A is a diagonalizable $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (not necessarily distinct; an eigenvalue of algebraic multiplicity m appears m times in this list) then the characteristic polynomial $f_A(t)$ is also given by the right-hand side of equation (1). In particular,

$$\begin{aligned} \det(A) \ = \ \det(A - tI)|_{t=0} \ &= \ f_A(0) \\ &= \ \text{constant term in } f_A(t) \\ &= \ \sigma_n(\lambda_1, \lambda_2, \ldots, \lambda_n) \\ &= \ \lambda_1 \lambda_2 \ldots, \lambda_n \ . \end{aligned}$$

Thus *the determinant of A is the product of its eigenvalues* (counted with multiplicity):

$$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n \ . \tag{2}$$

Furthermore, the coefficient of t^{n-1} in $f_A(t)$ is always $(-1)^{n-1}\mathrm{tr}(A)$.