Non-book problem for Assignment 1

NB 1.1. (a) Define binary operations "$+'$" and "$*$" on $\mathbf{R}$ by

$$x +' y \;=\; (x^{1/3} + y^{1/3})^3 \qquad \text{for all } x, y \in \mathbf{R}$$

and

$$a * x \;=\; a^3\, x \qquad \text{for all } a, x \in \mathbf{R}.$$

Check that $\mathbf{R}$, equipped with $+'$ as the addition operation, and $*$ as the scalar-multiplication operation (i.e. with $\mathrm{sm}(a, x) = a * x$), is a vector space. Also check that $x +' y$ does not equal $x + y$ unless $x = 0$, $y = 0$, or $y = -x$.

(b) Show the more general fact I asserted in class on Friday 1/17/25: Given any bijection $\phi : \mathbf{R} \to \mathbf{R}$, if we define operations $\mathrm{plus}_\phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $\mathrm{sm}_\phi : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by

$$\mathrm{plus}_\phi(x, y) \;=\; \phi\big(\phi^{-1}(x) + \phi^{-1}(y)\big)$$

and

$$\mathrm{sm}_\phi(a, x) \;=\; \phi\big(a\, \phi^{-1}(x)\big),$$

then the set $\mathbf{R}$, equipped with these operations, is a vector space. (Recall that for any function f, there is an inverse function if and only if f is a bijection, in which case "f^{-1}" denotes the inverse function [unless otherwise specified].)

Also check that the vector-space structure in part (a) is simply the one obtained this way from the bijection $x \mapsto x^{1/3}$ (i.e. the function $\phi : \mathbf{R} \to \mathbf{R}$ defined by $\phi(x) = x^{1/3}$). (By a *vector-space structure* on a set V, I simply mean a pair of operations

$$\mathrm{plus} : V \times V \to V \quad \text{and} \quad \mathrm{sm} : \mathbf{R} \times V \to V$$

such that V, equipped with "plus" as the addition operation and "sm" as the scalar-multiplication operation, is a vector space.)