Non-book problems for Assignment 8

Before starting, review the definition of "external" direct sum in the Direct Sums handout.

NB 8.1. Let W_1 and W_2 be finite-dimensional subspaces of a vector space V. In an earlier homework exercise (FIS 1.6/ 29(a)) you showed that $\mathsf{W}_1 + \mathsf{W}_2$ is finite-dimensional and that

$$\dim(\mathsf{W}_1 + \mathsf{W}_2) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2) - \dim(\mathsf{W}_1 \cap \mathsf{W}_2). \tag{1}$$

In this problem, you will give a different proof of formula (1).

Since W_1 and W_2 are vector spaces, the "external" direct sum $\mathsf{W}_1 \oplus_e \mathsf{W}_2$ is defined. (Note that $\mathsf{W}_1 \oplus_e \mathsf{W}_2$ is not the same vector space, or even the same *set*, as the subspace $\mathsf{W}_1 + \mathsf{W}_2 \subseteq \mathsf{V}$.) Let

$$\mathsf{K} = \{(v, -v) : v \in \mathsf{W}_1 \cap \mathsf{W}_2\} \subseteq \mathsf{W}_1 \oplus_e \mathsf{W}_2.$$

(a) Show that K is a subspace of $\mathsf{W}_1 \oplus_e \mathsf{W}_2$.

(b) Show that K is finite-dimensional and that $\dim(\mathsf{K}) = \dim(\mathsf{W}_1 \cap \mathsf{W}_2)$.

(c) Define $\mathsf{T} : \mathsf{W}_1 \oplus_e \mathsf{W}_2 \to \mathsf{W}_1 + \mathsf{W}_2$ by $\mathsf{T}(w_1, w_2) = w_1 + w_2$.
   (i) Show that T is linear.
   (ii) Show that $\mathsf{R}(\mathsf{T}) = \mathsf{W}_1 + \mathsf{W}_2$.
   (iii) Show that $\mathsf{N}(\mathsf{T}) = \mathsf{K}$.

(d) Recall from FIS exercise 1.6/ 25 that $\mathsf{W}_1 \oplus_e \mathsf{W}_2$ is also finite-dimensional and that

$$\dim(\mathsf{W}_1 \oplus_e \mathsf{W}_2) = \dim(\mathsf{W}_1) + \dim(\mathsf{W}_2). \tag{2}$$

Note also that $\mathsf{W}_1 \cap \mathsf{W}_2$ is finite-dimensional, since it's a subspace of W_1. (We could equally well have said "since it's a subspace of W_2.")

Using equation (2), earlier parts of this problem, and the Rank-Plus-Nullity Theorem, show that $\mathsf{W}_1 + \mathsf{W}_2$ is finite-dimensional, and prove the dimensional relation (1).

NB 8.2. In the previous problem, we assumed at the outset that both W_1 and W_2 were finite-dimensional. This assumption was more than was needed for several parts of the problem.

(a) For which parts of problem NB 8.1 could we have eliminated *all* assumptions of finite-dimensionality?

(b) For which parts of problem NB 8.1 would it have been enough to assume that *at least one* of W_1 and W_2 is finite-dimensional?

(c) Which parts of problem NB 8.1 actually used the assumption that *both* W_1 and W_2 are finite-dimensional?

NB 8.3. Let W_1 and W_2 be subspaces of a finite-dimensional vector space V. (Note that this is not the same hypothesis as in NB 8.1.) Show that if any two of the conditions (i), (ii), and (iii) are satisfied, then so is the third:

(i) $\mathsf{W}_1 \cap \mathsf{W}_2 = \{0_{\mathsf{V}}\}$.

(ii) $\mathsf{W}_1 + \mathsf{W}_2 = \mathsf{V}$.

(iii) $\dim(\mathsf{W}_1) + \dim(\mathsf{W}_2) = \dim(\mathsf{V})$.

Hence if any two of (i)–(iii) are satisfied, then V is the "internal" direct sum of W_1 and W_2. (This problem is an extension of FIS 1.6/29(b), in which you proved "(ii) + (iii) $\implies$ (i).")