Polynomials and Polynomial Functions

version date: 1/25/2025

In these notes, "FIS" is the textbook *Linear Algebra* by Friedberg, Insel, and Spence, 5th edition.

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1 Introduction

Polynomials and *polynomial functions*, while closely related, are not quite the same thing. In FIS, this is alluded to briefly (and only implicitly) on p. 10, after a definition of "polynomial" that is actually imprecise. In courses up through calculus, the distinction is not really important, but it becomes important in many higher-level courses.

In abstract algebra, polynomials are regarded as *formal expressions*—not functions—that can be added and multiplied according to certain rules that are *motivated* by the behavior of polynomial *functions*. (In these notes, the rules for multiplying two general polynomials are irrelevant; the only multiplication of relevance to us will be the multiplication of a polynomial by a scalar.) For clarity in this discussion, I'll often refer to these as "abstract polynomials", but this is *not* standard terminology.

In the abstract polynomial $1+2x-3x^2$, the letter x is called an *indeterminate*, rather than a variable. We speak of (abstract) polynomials *in a given indeterminate*[1], and the set of all such (abstract) polynomials with coefficients in a specific field[2] is given a name that includes the *name(s)* of the field and the indeterminate. For the set of polynomials in the indeterminate x with real coefficients,we may use the (standard!) notation "$\mathbf{R}[x]$", or non-standard notation notation selected by an author or instructor, such as "$P(\mathbf{R}; x)$".

The expressions "$1 + 2x - 3x^2$" and "$1 + 2y - 3y^2$" are abstract polynomials in two different indeterminates, x and y, hence are not the same abstract polynomial. But they determine the same polynomial <u>*function*</u> $f : \mathbf{R} \to \mathbf{R}$, namely $t \mapsto 1 + 2t - 3t^2$. [3]

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[1] In these notes, "polynomial" means "polynomial in a single indeterminate". (There is such a thing as polynomials in two or more indeterminates; we're just not talking about them here.)

[2] Or with coefficients in something called a *commutative ring*, which is more general than a field. You can think of a commutative ring as a field except for lacking the property that all nonzero elements have a multiplicative inverse. An example is the ring of integers, $\mathbf{Z}$. However, in this class, we address only polynomials with coefficients in a *field*.

[3] Said another way: having introduced $P(\mathbf{R}; x)$ and $P(\mathbf{R}; y)$—the sets of real-coefficient polynomials in *non-variable* indeterminates specifically denoted x and y—if we then allow ourselves to *change the meaning* of " x" or "y" above to "*dummy <u>variable</u>* used for writing down a formula for function-values," then the functions from $\mathbf{R}$ to $\mathbf{R}$ defined by $x \mapsto 1 + 2x - 3x^2$ and $y \mapsto 1 + 2y - 3y^2$ are *identical*. Be warned that **changing the meaning of notation mid-discussion is <u>generally</u> a bad idea.** But in this instance, this generally-bad idea does a good job of reproducing the *thought process* we're using to associate a polynomial with a function, so *for the purposes of writing down a function determined by a polynomial*, we usually allow this notational flexibility (outside these notes).

For the remainder of these notes, the word *polynomial(s)*, when not followed by the word *function(s)*, means *abstract* polynomial(s).

2 Some notation and terminology

To maintain the distinction between polynomials with polynomial *functions*, in these notes we use the notation $P(\mathbf{R}; x)$ for the set of polynomials in the indeterminate x (etc. for any other indeterminate) and $P(\mathbf{R})$ for the set of corresponding polynomial functions (see Section 2.2). For $n \geq 0$, we write $P_n(\mathbf{R}; x)$ for the set of polynomials of degree at most n, in the indeterminate x. When equipped with certain "standard" operations of addition and scalar multiplication, the sets $P(\mathbf{R}; x)$ and $P(\mathbf{R})$ become vector spaces, so we will often refer to these sets as *spaces*. When we call these sets *spaces*, the "standard operations"—which we have not specified yet—are assumed.

There is only *one* space $P(\mathbf{R})$ of polynomial *functions* from $\mathbf{R}$ to $\mathbf{R}$. However, there are *infinitely* many spaces of abstract polynomials, one for every conceivable name we could choose for an indeterminate. For example, since the letter x and letter y are not the same, the spaces $P(\mathbf{R}; x)$ and $P(\mathbf{R}; y)$ are not the same. However, these spaces are "equivalent" in a sense we will not make precise in this version of these notes. For simplicity, in these notes I will use only the space $P(\mathbf{R}; x)$—i.e. when talking about abstract polynomials, I will always use "x" as the indeterminate—except when making comparisons like "$P(\mathbf{R}; x)$ vs. $P(\mathbf{R}; y)$". But anything we state about polynomials in x has an immediate translation into a corresponding statement about polynomials in y or any other indeterminate.

Terminological simplification. As has already been done above, we let ourselves use the term "polynomial in x" for "polynomial in the indeterminate x" (etc. for other indeterminates).

Remark 2.1 The notation "$P(\mathbf{R}; x)$" is something I've **cooked up** as an alternative to the standard notation "$\mathbf{R}[x]$", just to stay closer to the notation "$P(\mathbf{R})$" that FIS uses for both the space of polynomials functions and the space of polynomials in the indeterminate x.[4] (Until Chapter 5, FIS takes the letter x to be the *fixed* indeterminate for all abstract polynomials, even when the same letter is used for many other things on the same or nearby pages. Many examples and exercises in FIS make no sense without an agreement—which FIS leaves implicit—that "this is always what 'x' means when we're talking about polynomials." This is very similar to the way 'x' is used as the variable for most functions in Calculus 1 and lower-level courses, but it can cause difficulties in our course.) A bonus is that "$P(\mathbf{R}; x)$", like

[4]Although the book's notation usage of "$P(\mathbf{R})$", and its notation for individual polynomials,' is ambiguous and somewhat misleading, this usage is not uncommon. In a different course, I might not mind the ambiguity, but in MAS4105 I find that it interferes with getting across certain ideas.

"$P(\mathbf{R})$", is easier to modify than is "$\mathbf{R}[x]$" when we want to incorporate the restriction "of degree at most n" into the notation.

2.1 Notation for individual polynomials

For individual abstract polynomials in x, some textbook-authors use notation of the form "$f(x)$", while others use notation such as "f" that does not incorporate x. **only if the author does not use standard notation for functions,** in which "$f(x)$" denotes the output of a function f when the input is x. (Abstract polynomials don't have inputs or outputs.) The FIS textbook uses notation such as f for *most* functions, but uses $f(x)$ for *polynomial* functions, as well as for abstract polynomials. This inconsistency can cause confusion, especially among students who are still getting used to calling functions "f" rather than "$f(x)$."

To avoid this potential confusion, in these notes we opt for *consistency* over what students may find more familiar and comfortable. Notation of the form "$f(x)$" will be used exclusively for *outputs of functions*. [5] Parenthesis-free names will be used for functions and for abstract polynomials.

For example, we will give a name such as f or p (**not** $f(x)$ or $p(x)$) to the abstract polynomial $1 + 2x - 3x^2 \in P(\mathbf{R}; x)$. If we want to talk about the analogous polynomial $1 + 2y - 3y^2 \in P(\mathbf{R}; y)$ at the same time, we need a different name, e.g. q (**not** $q(y)$). If we're interested instead in polynomial *functions* from $\mathbf{R} \to \mathbf{R}$, we could give a name like f to the function $x \mapsto 1+2x-3x^2$, in which case $f(x) = 1+2x-3x^2$, $\quad f(y) = 1+2y-3y^2$, $f(t) = 1 + 2t - 3t^2$, etc. (In the last sentence, we have assumed that x, y, and t have not been set aside as names of indeterminates.) Remember that for a *function*, **the letter chosen to represent the domain-variable is not part of the name of the function.**

2.2 Functions associated with abstract polynomials

Suppose $p \in P(\mathbf{R}; x)$ is a polynomial. Then, by definition, there is a non-negative integer n (the *degree* of f) and unique numbers $a_0, a_1, a_2, \ldots, a_n$ such that

$$p = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n, \tag{2.1}$$

and such that, if $n > 0$, then $a_n \neq 0$.[6] **In these notes**, we will use the notation $\hat{p}$ for the associated function from $\mathbf{R}$ to $\mathbf{R}$, i.e. the function obtained by replacing the indeterminate

[5] Of course, there are other uses of parentheses in mathematics. For example, in "$A(B+C)$" could mean several different things, depending on what A, B, and C are. If these objects are (say) real numbers or algebraic expressions, then "$A(B+C)$" means the product of the numbers/expressions A and $B+C$. But if we are given that A is a *function* whose domain is a set S on which a binary operation "+" is defined, and B and C are elements of S, then $A(B+C)$ means the output of the function A when the input is $B+C$. It is always the writer's responsibility to make sure that, whenever he/she is using parentheses, only one interpretation makes sense.

[6] For "clarity of pattern", we have included a_1, a_2, a_1x, and a_2x^2 in the preceding sentence. However, a_1 and a_1x are present only if $n \geq 1$, and a_2 and a_2x^2 are present only if $n \geq 2$.

x by the domain-variable of a function from $\mathbf{R}$ to $\mathbf{R}$. In other words, $\hat{p} : \mathbf{R} \to \mathbf{R}$ is the function defined by

$$\hat{p}(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n. \tag{2.2}$$

Definition 2.2 A function $g : \mathbf{R} \to \mathbf{R}$ is called a *polynomial function* if $g = \hat{p}$ for some $p \in P(\mathbf{R}; x)$. In these notes, $P(\mathbf{R})$ denotes the set of all polynomial functions from $\mathbf{R} \to \mathbf{R}$.

Recall that $\mathcal{F}(\mathbf{R}, \mathbf{R})$ denotes the space of all functions from $\mathbf{R}$ to $\mathbf{R}$. Thus we can also write the definition of $P(\mathbf{R})$ either of the following ways:

$$\begin{aligned} P(\mathbf{R}) &= \{g \in \mathcal{F}(\mathbf{R}, \mathbf{R}) : \text{there exists } p \in P(\mathbf{R}; x) \text{ such that } g = \hat{p}\ \} && (2.3) \\ &= \{\hat{p}\ :\ p \in P(\mathbf{R}; x)\}\ \subseteq\ \mathcal{F}(\mathbf{R}, \mathbf{R}). && (2.4) \end{aligned}$$

Exercise 2.3 Show that $P(\mathbf{R})$ is a subspace of $\mathcal{F}(\mathbf{R}, \mathbf{R})$.

When we refer to "the vector space $P(\mathbf{R})$" (or simply "the space $P(\mathbf{R})$", the operations are always understood to be the ones inherited from $\mathcal{F}(\mathbf{R}, \mathbf{R})$; i.e. **we are always implicitly regarding $P(\mathbf{R})$ as a subspace of $\mathcal{F}(\mathbf{R}, \mathbf{R})$.**

3 Other differences between polynomials and polynomial functions

In FIS, "polynomial" sometimes means abstract polynomial and sometimes means polynomial function., and uses the notation "$P(\mathbf{R})$" both for the space $P(\mathbf{R}; x)$ and the space called $P(\mathbf{R})$ in these notes (the space of polynomial functions). In a later section of these notes [**not yet written** as of 1/25/2025 because we haven't introduced the relevant ideas and terminology in our class yet], we'll be able to show that these two vector spaces are "essentially the same". However, FIS treats them as *being* the same with no careful justification.[7]

One fundamental difference between (abstract) polynomials and polynomial functions is the definition of *equality*. **By definition**, two abstract polynomials in x, say $p = a_0 + a_1 x + \cdots + a_n x^n$ and $q = b_0 + b_1 x + \cdots + b_m x^m$, are equal if and only if $n = m$ and $a_i = b_i$ for each $i \in \{0, \ldots, n\}$, but **by definition**, two functions $g, h : \mathbf{R} \to \mathbf{R}$ are equal if and only if $g(t) = h(t)$ for all $t \in \mathbf{R}$. The criteria for equality are not the same.

This begs an important question: is it possible for two different (abstract) polynomials in x—possibly even polynomials of different degree—to have the same associated

[7]There is a very quiet reference to the issue on p. 10, which refers the reader to p. 564 the *very last page* of the *very last appendix* of the book. Even on p. 564, only *partial* justification for regarding $P(\mathbf{R}; x)$ and $P(\mathbf{R})$ as being "the same" is given.

function in $P(\mathbf{R})$? I.e. for p and $q \in P(\mathbf{R}; x)$, is it possible to have $p \neq q$ and yet have $\hat{p} = \hat{q}$?

Fortunately, the answer to this question is *no.*[8]. One way to see this is that if two infinitely differentiable functions $g, h \in \mathcal{F}(\mathbf{R}, \mathbf{R})$ are equal, then and $g^{(k)}(t) = h^{(k)}(t)$ for all $k \geq 0$ adn all $t \in \mathbf{R}$ (where the superscript "(k)" denotes k^{th} derivative for $k \geq 1$, and where we define $f^{(0)} = f$ for every $f \in \mathcal{F}(\mathbf{R}, \mathbf{R})$). In particular this applies with $g = \hat{p}$ and $h = \hat{q}$, where $p, q \in P(\mathbf{R}; x)$ (since, from Calculus 1, polynomial functions are infinitely differentiable). But if $p = a_0 + a_1 x + \cdots + a_n x^n$, then

$$\hat{p}^{(k)}(0) = \begin{cases} k!\, a_k & \text{if } 0 \leq k \leq n, \\ 0 & \text{if } k > n. \end{cases}$$

Similarly if $q = b_0 + b_1 x + \cdots + b_m x^m$, then

$$\hat{q}^{(k)}(0) = \begin{cases} k!\, b_k & \text{if } 1 \leq k \leq m \\ 0 & \text{if } k > m. \end{cases}$$

It follows that if $\hat{p} = \hat{q}$ then $n = m$ and $a_i = b_i$ for each $i \in \{1, \ldots, n\}$, and hence $p = q$.

We record this just-proved fact as the following proposition:

Proposition 3.1 *Let $T : P(\mathbf{R}; x) \to P(\mathbf{R})$ be the function defined by $T(p) = \hat{p}$* (see equations (2.1) and (2.2)). *Then T is one-to-one.* ■

Corollary 3.2 *The function T in Proposition 3.1 is a bijection from $P(\mathbf{R}; x)$ to $P(\mathbf{R})$.* (Recall that a bijection is function that is both *injective* [one-to-one] and *surjective* [onto].)

Proof: Proposition 3.1 shows that T is one-to-one. The very *definition* of the set $P(\mathbf{R})$ shows that T is onto. (Recall Definition 2.2 or equation (2.3) or (2.4).) ■

The vector-space operations on $P(\mathbf{R}; x)$ and $P(\mathbf{R})$ are also defined quite differently. Whereas the operations on $P(\mathbf{R}; x)$ are defined *termwise*[9] while the operations on $P(\mathbf{R})$, a space of functions, are defined *pointwise*[10].

TO BE CONTINUED.

[8]However, the answer would be "yes" if **R** were replaced by a finite field, an object we have not discussed, but that FIS *does* discuss.

[9]Meaning that for addition, coefficients of like powers of x are added; for scalar multiplication by c, the coefficient of each power of x is multiplied by c.

[10]I.e. by using addition and scalar-multiplication of function *values* (outputs) at each point of the domain (each input).