MAA 4211, Fall 2001—Homework # 5 non-book problems

Hand in B1, B2ab, and B3. Do not do B2 and B3 until you have read the "Interiors, Closures, and Boundaries" handout.

B1. Let $E = \mathbf{Q}$ with the usual metric ($d(x,y) = |x-y|$). Give an example of a nonempty, proper subset of $\mathbf{Q}$ that is both open and closed.

B2. (Hand in only parts (a) and (b).) Let $E = E^2$ (Euclidean 2-space). Let $p \in E$ and let $r > 0$.

(a) Show that $\overline{B_r(p)} = \overline{B}_r(p)$ (i.e. the closure of an open ball is the closed ball with the same center and radius).

(b) Show that $\partial B_r(p)$ is the *sphere* of radius r centered at p, defined as $\{q \in E \mid d(p,q) = r\}$. (This is the general definition of "sphere" for an arbitrary metric space; spheres in E^2 are circles.)

(c) Re-do parts (a) and (b) with E^2 replaced by E^n, where n is arbitrary. Once (a) and (b) are done, you should find this easy; if not, then your arguments in (a) and (b) are probably wrong.

B3. Give an example of a metric space E for which there is an open ball $B_r(p)$ ($r > 0$) whose closure is *not* the closed ball $\overline{B}_r(p)$. (Hint: you have already come across a metric space with this property.)

B4. (Do not hand in.) Let S be an arbitrary set, let P be a property that a given <u>subset</u> of S may or may not have, and let Q be a property that a given <u>element</u> of S may or may not have. Make sure you understand that P and Q are entirely different sorts of animals.

Think about why the following two statements about arbitrary subsets $X \subset S$ are *not* equivalent in general (nor is there even a one-way implication in either direction; the statements are simply unrelated).

- Statement 1: "The subset X does not have property P."
- Statement 2: "The complement of X *does* have property P."

Experiment with some examples of S, X, and P until there is no doubt in your mind as to why Statements 1 and 2 are not logically equivalent. (E.g consider $S = \mathbf{R}$, $P =$ the property of being a finite set.) Once you are comfortable with this, think about the case in which S is a metric space and P is either of the properties "open" or "closed". Make sure you understand that there is no contradiction in the fact that Statements 1 and 2 are not equivalent (in general), while the following two statements *are* equivalent:

- Statement 3: "$Y = \{y \in S : y \text{ does not have property } Q\}$."
- Statement 4: "The complement of Y is $\{y \in S : y \text{ *does* have property } Q\}$."