MAA 4211, Fall 2001—Homework # 7 non-book problems

Do not hand these in.

C1. Let $E = (0,1) \times (0,1) = \{(x,y) \in \mathbf{R}^2 \mid 0 < x < 1 \text{ and } 0 < y < 1\}$, and give E the Euclidean metric. Prove that for any $p \in E$, the complement of p is arcwise connected, and hence connected.

C2. Let E be as in the previous problem. Prove that no continuous real-valued function on E can be injective. (Hint: There is a reason why I had you do problem 1 first.) Remark: Continuity is essential in this statement. A long time ago we saw that E and the interval $(0,1)$ have the same cardinality, so there *is* a bijective function from E to $(0,1)$ (hence an injective function from E to $\mathbf{R}$—it's just that such a function can't be continuous.

Definition. Let E be a metric space and let $p \in E$. The *path component of* p (or the *path component of* E *containing* p) is the set

$$\{q \in E \mid \exists \text{ a continuous function} f : [0,1] \to E \text{ with} f(0) = p \text{ and} f(1) = q\}.$$

A *path component of* E is any set which is the path component of some point in E. (Note that, by this definition, a path component is never empty. Also note that a metric space is arcwise connected if and only if it has exactly one path component, i.e. if the path component of any point is the entire metric space).

C3. Prove that every metric space is the disjoint union of its path components.

C4. Let A be an open subset of $\mathbf{E}^n$, and give A the Euclidean metric inherited from $\mathbf{E}^n$. Prove that every path component of A is an open subset of $\mathbf{E}^n$.