MAA 4211, Fall 2014—Assignment 4's non-book problems

B1. Let $(E,d) = E^2$ (Euclidean 2-space). Let $p \in E$ and let $r > 0$.

(a) Show that $\overline{B_r(p)} = \overline{B}_r(p)$ (i.e. the closure of an open ball is the closed ball with the same center and radius).

(b) Show that $\partial B_r(p)$ is the *sphere* of radius r centered at p, defined as $\{q \in E \mid d(p,q) = r\}$. (This is the general definition of "sphere" for an arbitrary metric space; spheres in E^2 are circles.)

(c) Re-do parts (a) and (b) with E^2 replaced by E^n, where n is arbitrary. Once (a) and (b) are done, you should find this easy; if not, then your arguments in (a) and (b) are probably wrong.

B2. Give an example of a metric space E in which there is an open ball $B_r(p)$ whose closure is *not* the closed ball $\overline{B}_r(p)$. (You have already seen at least two metric spaces with this property.)

B3. (a) Let (E,d) be a metric space, $p \in E$, $r > 0$. Let $S_r(p)$ denote the sphere of radius r centered at p (see B1(b)). Prove that $\partial(B_r(p)) \subset S_r(p)$.

(b) Give an example of a metric space (E,d) in which there is an open ball $B_r(p)$ for which $\partial(B_r(p)) \neq S_r(p)$.

B4. Let d_1 and d_2 be two metrics on a nonempty set E. Call a set $S \subset E$ "d_1-open" if it is open in the metric space (E,d_1), and "d_2-open" if it is open in the metric space (E,d_2). Analogously define "d_i-bounded set" and "d_i-convergent sequence".

(a) Suppose that there exists $c > 0$ such that $d_2(p,q) \leq c d_1(p,q)$ for all $p,q \in E$. Prove that every d_2-open subset of E is d_1-open.

(b) Metrics d_1, d_2 on E are called *equivalent* if Suppose that there exist $c_1, c_2 > 0$ such that for all $p,q \in E$, $d_2(p,q) \leq c_1 d_1(p,q)$ and $d_1(p,q) \leq c_2 d_2(p,q)$. Prove the following:

(i) Writing "$d_1 \sim d_2$" for "d_1, d_2 are equivalent metrics", show that $\sim$ is an equivalence relation on the set of all metrics on E.

(ii) Equivalent metrics determine the same open sets and the same closed sets. I.e. if d_1 and d_2 are equivalent and $U \subset E$, then U is d_1-open iff U is d_2-open, and U is d_1-closed iff U is d_2-closed.

(iii) Equivalent metrics determine the same bounded sets. I.e. if d_1 and d_2 are equivalent and $U \subset E$, then U is d_1-bounded if U is d_2-bounded.

(iv) Equivalent metrics determine the same convergent sequences and the same limits of convergent sequences. I.e. if d_1 and d_2 are equivalent, $q \in E$, and $\{p_n\}$ is a sequence in E, then $\{p_n\}$ is d_1-convergent to q iff $\{p_n\}$ is d_2-convergent to q.

B5. Let $\| \ \|$ and $\| \ \|'$ be two norms on a vector space V. We call these two norms *equivalent* if there exist $c_1, c_2 > 0$ such that for all $v \in V$, $\|v\| \leq c_1 \|v\|'$ and $\|v\|' \leq c_2 \|v\|$. Prove that if norms $\| \ \|$ and $\| \ \|'$ are equivalent, then their associated metrics are equivalent.

B6. Let $n \in \mathbf{N}$. Prove that the ℓ^1, ℓ^2, and ℓ^∞ norms on $\mathbf{R}^n$ are all equivalent to each other (i.e. each is equivalent to the other two), and hence that their associated metrics are equivalent to each other.

B7. Let (E, d) be a metric space, let $\{p_n\}_{n=1}^{\infty}$ be a sequence in E, and define sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ by

$$\begin{aligned} x_n &= p_{2n-1} \quad \forall n \in \mathbf{N}, \\ y_n &= p_{2n} \quad \forall n \in \mathbf{N}. \end{aligned}$$

(In other words, $\{x_n\}$ and $\{y_n\}$ are the subsequences of $\{p_n\}$ given by the odd-numbered terms and even-numbered terms, respectively.) Prove that the following are equivalent:

(i) $\{p_n\}_{n=1}^{\infty}$ converges.

(ii) Both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge, and their limits are equal.

Prove also that if condition (ii) holds, then $\lim_{n\to\infty} p_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

B8. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers, and let $\alpha = \limsup_{n\to\infty} a_n$.

(a) Prove that there exists a subsequence of $\{a_n\}$ that converges to α.

(b) Prove that α is the largest real number with the property in part (a). I.e. prove that if $c > \alpha$, then there exists no subsequence of $\{a_n\}$ converging to c.

(c) State the analogs of parts (a) and (b) for "lim inf". Do not write out the analogous proofs, but summarize briefly what changes would be required in your proofs of (a) and (b) to prove these analogs.

B9. Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of real numbers.

(a) Prove that

$$\limsup_{n\to\infty} (a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

(b) Give an example that shows that the inequality in part (a) can be strict.