MAA 4211, Fall 2014—Assignment 6's non-book problems

B1. Let (E, d) be a metric space, $S \subset E$. For purposes of this problem and the next, call S *non-connected* if S is not connected.[1] Prove that S is non-connected if and only if $S = A \bigcup B$ for some nonempty sets $A, B \subset S$ for which $\overline{A} \bigcap B = \emptyset = A \bigcap \overline{B}$. (Here $\overline{A}$ and $\overline{B}$ denote the closures of A and B in E, not in the subspace S.)

B2. Let (E, d) be a metric space, $S \subset E$ a nonempty subset, and $p \in E$. Recall from the first midterm that the *distance from p to S*, written $\text{dist}(p, S)$, is defined to be $\inf\{d(p, q) \mid q \in S\}$. On that exam you were asked to prove that $\text{dist}(p, S) = 0$ if and only if $p \in \overline{S}$. You may assume that fact here (there is a proof in the first-midterm solutions).

Using the result of B1, prove that S is non-connected if and only if $S = A \bigcup B$ for some nonempty sets $A, B \subset S$ for which every point of each set is a positive distance from the other set (i.e. $\text{dist}(p, B) > 0 \ \forall p \in A$ and $\text{dist}(p, A) > 0 \ \forall p \in B$).

Motivation for this problem: Recall that, heuristically, we wanted "S is not connected" to mean that S cannot be partitioned into two nonempty disjoint subsets that "don't touch each other". There is no official definition of one subset of a metric space *touching*, or not touching, another. However, were we (not unreasonably) to define "A does not touch B" to mean "every point of A is a positive distance from B", then the characterization of non-connectedness in this problem would turn the heuristic characterization of "not connected" into a precise one that agrees with the mathematical definition.

B3. Let (E, d) be a metric space.

(a) Let $p \in E$. Show that the singleton set $\{p\}$ is connected.

(b) Let $p \in E$, and let $\mathcal{F}_p = \{S \subset E \mid S \text{ is connected and } p \in S\} \subset P(E)$. Let

$$C_p = \bigcup_{S \in \mathcal{F}_p} S.$$

Prove that C_p is connected.

(Do not re-invent the wheel to prove this. You should need no more than a couple of sentences, if you apply a couple of facts already proven.)

The set C_p defined above is called the *connected component of p in E* (or in (E, d)). We will use the notation "C_p" with this meaning for the rest of this problem. A subset $C \subset E$ is called a *connected component* of E if $C = C_p$ for some $p \in E$.

(c) For $p \in E$, prove that C_p is the largest connected set containing p, in the following sense: if $S \subset E$ is connected and $p \in S$, then $S \subset C_p$.

(d) Define a relation $\sim$ on E by declaring $p \sim q$ if and only if $q \in C_p$. Prove that $\sim$ is an equivalence relation, and that the equivalence classes are exactly the connected components of E.

[1] Although it is tempting to use the term "disconnected" for "not connected", topologists generally don't do this, instead reserving "disconnected" as one piece of the terminology for topological (sub)spaces that fail in some spectacular way to be connected, such as *totally disconnected* spaces (see problem B3(e)). The most common terminology for "not connected" is "not connected", not "non-connected". In this problem I'm using "non-connected" because "S is not connected if and only if ..." could be misinterpreted.

Recall that, for any equivalence relation on a set S, the equivalence classes partition S into pairwise disjoint subsets. (For the relation above, "pairwise disjointness" means that for any $p, q \in E$, either $C_p = C_q$ or $C_p \bigcap C_q = \emptyset$.) Thus a metric space is always the disjoint union of its connected components.

(e) (E, d) is called *totally disconnected* if the only nonempty connected subsets of E are the singleton sets. Prove that $\mathbf{Q}$, with its usual metric, is totally disconnected.

Note: in Assignment 3, Problem B3, you effectively were proving that $\mathbf{Q}$ is not connected (but "connected" was not in our mathematical vocabulary at the time). Now you are proving something much stronger.

(f) Prove that every connected component of (E, d) is a closed subset of E. (Here (E, d) is a general metric space again, not totally disconnected.)

(g) Use part (f) to prove that if (E, d) has only finitely many connected components, then each connected component is both open and closed.

B4. (Same as problem B3 on Assignment 5, but done another way.) Let (E, d) be a metric space, let $\{p_n\}_{n=1}^{\infty}$ be a convergent sequence in E, let $p = \lim_{n\to\infty} p_n$, let $N \in \mathbf{N}$, and let $\epsilon > 0$. Assume that for all $n, m \geq N$, $d(p_n, p_m) < \epsilon$. Use the appropriate half of the "sequential characterization of continuity" to prove that $d(p_N, p) \leq \epsilon$. *Hint*: Consider the function $f : E \to \mathbf{R}$ defined by $f(q) = d(p_N, q)$.

B5. In problem #3 on p. 91 of Rosenlicht, suppose you remove the hypothesis that the sets S_1, S_2 are both closed. Is the conclusion still true? (Prove your answer, of course.)