MAA 4211, Fall 2015—Assignment 1's non-book problems

B1. Let X, Y, and Z be sets and let $f : X \to Y,\ g : Y \to Z$ be functions.

(a) Show that if f and g are bijective, then so is $g \circ f$.

(b) Suppose $g \circ f$ is bijective. Does it follow that f and g are bijective? If your answer is "yes", prove it; if "no", give a counterexample.

B2. Define $f : \mathbf{R} \to \mathbf{R}^2$ by $f(x) = (\cos x, \sin x)$. Let

$$D = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 < 1\}$$

and

$$S = \{(x, y) \in \mathbf{R}^2 \mid x^2 + (y-1)^2 < 1\}.$$

Identify (a) $f^{-1}(S)$, (b) $f^{-1}(D)$, and (c) $f^{-1}(f((-\frac{\pi}{2}, \frac{\pi}{2})))$.

B3. Find a formula expressing the cardinality of the cartesian product of two finite sets X, Y in terms of the cardinalities of X and Y. Justify your answer.

B4. Recall that the *power set* of a set X is the set $P(X)$ of all subsets of X (an *element* of $P(X)$ is a *subset* of X). For every set X, let 2^X denote the set of all functions from X to the two-element set $\{0, 1\}$.

(a) Let X be a nonempty set. Exhibit a bijection $P(X) \to 2^X$ (or $2^X \to P(X)$). Thus $P(X)$ has the same cardinality as 2^X.

(b) If X is a finite nonempty set of cardinality n, what is the cardinality of 2^X? (Derive your formula; don't just write down the answer.) If your answer is correct, you will see where the notation "2^X" comes from.

In view of part (a), part (b) gives the cardinality of $P(X)$ for a finite nonempty set.

B5. Show that every subset of a countable set is countable.

B6. Prove that every infinite set has a countably infinite subset.

B7. Prove that a nonempty set X is countable if and only if there exists a surjective map $\mathbf{N} \to X$.

B8. Prove that a countable union of countable sets countable; i.e., if $\{A_i\}_{i \in I}$ is a collection of sets, indexed by $I \subset \mathbf{N}$, with each A_i countable, then $\bigcup_{i \in I} A_i$ is countable. *Hints*: (i) Show that it suffices to prove this for the case $I = \mathbf{N}$. (ii) By Problem B7, for each $i \in \mathbf{N}$ there is a surjective map $f_i : \mathbf{N} \to A_i$. Use these maps to produce a surjective map $\mathbf{N} \times \mathbf{N} \to \bigcup_{i \in \mathbf{N}} A_i$, and then use earlier results to conclude that $\bigcup_{i \in \mathbf{N}} A_i$ is countable.