MAA 4211, Fall 2015—Assignment 3's non-book problems

B1. Let $a, b \in \mathbf{R}, a < b$. Prove that the interval $(a, b] := \{x \in \mathbf{R} \mid a < x \leq b\}$ is not an open set. (We proved this informally in class; you should write out a careful proof.) B2.

Let $n \geq 1$ and let $\mathbf{E}^n$ denote Euclidean n-space. Let $p \in \mathbf{E}^n$, $r \geq 0$. Let $\overline{B}_r(p)$ denote the closed ball of radius r centered at p. Prove that $\overline{B}_r(p)$ is not an open set.

Remember: (i) "Closed" does not imply "not open". The fact that a closed ball in a metric space is a closed set does not imply that a closed ball can't be an open set. (ii) There is no such thing as "proof by picture". If you are asserting, for example, that a certain open ball contains points of some other set, you have to *prove* that assertion, not merely assert that it's true based on some picture you've drawn and your intuition.

B3. Define a metric d on the set of rational numbers $\mathbf{Q}$ by $d(x, y) = |x-y|$ (the restriction to $\mathbf{Q}$ of the standard metric on $\mathbf{R}$). Give an example, with proof, of a nonempty, proper subset of $(\mathbf{Q}, d)$ that is both open and closed in this metric space. (Do not expect your subset to be either open or closed in $\mathbf{R}$, let alone *both* open and closed in $\mathbf{R}$. There is no nonempty, proper subset of $\mathbf{R}$ that is both open and closed with respect to the standard metric.)

B4. Let (E, d) be a metric space. For purposes of this problem, for each $p \in E$ define a property we'll call "boundedness with respect to p" as follows: a set $S \subset E$ is *bounded with respect to* p if S is contained in some ball *centered at* p. Show that the following are equivalent, for every $S \subset E$:

(i) S is bounded with respect to p.

(ii) S is bounded.

(iii) S is bounded with respect to q for all $q \in E$.

B5. Let $(V, \| \ \|)$ be a normed vector space, with the associated metric. Prove the following corollary of problem B4: A set $S \subset V$ is bounded if and only if there exists $M \in \mathbf{R}$ such that $\|v\| \leq M$ for all $v \in S$.

Write down concretely what this corollary is saying in the case $(V, \| \ \|) = (\mathbf{R}, | \ |)$, and compare with the definition of "bounded sequence" given for problem 1b on the main Assignment 3 webpage.

B6. Let $(E, d) = \mathbf{E}^2$ (Euclidean 2-space). Let $p \in E$ and let $r > 0$.

(a) Show that $\overline{B_r(p)} = \overline{B}_r(p)$ (i.e. the closure of an open ball is the closed ball with the same center and radius).

(b) Show that $\partial B_r(p)$ is the *sphere* of radius r centered at p, defined as $\{q \in E \mid d(p, q) = r\}$. (This is the general definition of "sphere" for an arbitrary metric space; spheres in $\mathbf{E}^2$ are circles.)

(c) Re-do parts (a) and (b) with $\mathbf{E}^2$ replaced by $\mathbf{E}^n$, where n is arbitrary. Once (a) and (b) are done, you should find this easy; if not, then your arguments in (a) and (b) are probably wrong.

B7. Give an example of a metric space E in which there is an open ball $B_r(p)$ whose closure is *not* the closed ball $\overline{B}_r(p)$.

B8. (a) Let (E, d) be a metric space, $p \in E$, $r > 0$. Let $S_r(p)$ denote the sphere of radius r centered at p (see B6(b)). Prove that $\partial(B_r(p)) \subset S_r(p)$.

(b) Give an example of a metric space (E, d) in which there is an open ball $B_r(p)$ for which $\partial(B_r(p)) \neq S_r(p)$.