MAA 4211, Fall 2015—Assignment 4's non-book problems

B1. Let (E, d) be a metric space, let $\{p_n\}_{n=1}^{\infty}$ be a sequence in E, and define sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ by

$$
\begin{aligned}
x_n &= p_{2n-1} \quad \forall n \in \mathbf{N}, \\
y_n &= p_{2n} \quad \forall n \in \mathbf{N}.
\end{aligned}
$$

(In other words, $\{x_n\}$ and $\{y_n\}$ are the subsequences of $\{p_n\}$ given by the odd-numbered terms and even-numbered terms, respectively.) Prove that the following are equivalent:

(i) $\{p_n\}_{n=1}^{\infty}$ converges.

(ii) Both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge, and their limits are equal.

Prove also that if condition (ii) holds, then $\lim\limits_{n \to \infty} p_n = \lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} y_n$.