MAA 4211, Fall 2015—Assignment 5's non-book problems

B1. (a) Let E be a nonempty set. Show that the relation "d_1 is equivalent to d_2", on the set of metrics on E, is transitive. (This completes the proof that this relation is an equivalence relation. In class we observed that this relation is clearly reflexive and symmetric.)

(b) Let V be a vector space. Show that the relation "$\| \ \|_1$ is equivalent to $\| \ \|_2$", on the set of norms on V, is transitive. (This completes the proof that this relation is an equivalence relation.)

B2. Let $n \in \mathbf{N}$. Recall that the ℓ^1-norm $\| \ \|_1$ on $\mathbf{R}^n$ is defined by $\|(x_1, x_2, \ldots, x_n)\| = \sum_{i=1}^n |x_i|$. Prove that the norms $\| \ \|_1$ and $\| \ \|_\infty$ (the ℓ^∞ norm) are equivalent norms on $\mathbf{R}^n$, and hence that their associated metrics are equivalent.

Remark: In class we proved that the ℓ^2-norm and ℓ^∞ norm on $\mathbf{R}^n$ are equivalent. Therefore problems B1 and B2 imply that the ℓ^1-norm and ℓ^2-norm on $\mathbf{R}^n$ are equivalent, and hence so are their associated metrics.

B3. Let E be a nonempty set. Given two equivalent metrics d_1 and d_2 on E, and $i \in \{1, 2\}$, let us say that a sequence $(p_n)_{n=1}^\infty$ is d_i-*Cauchy* if it is Cauchy as a sequence in (E, d_i), that it is d_i-*bounded* if it is bounded as a sequence in (E, d_i), etc. for other adjectives that may apply to sequences. If a sequence $(p_n)_{n=1}^\infty$ converges in (E, d_i), in this problem write "$\lim^{(d_i)}_{n\to\infty} p_n$" for the value of the limit.

In class, the following Proposition was stated, with part (a)(ii) accidentally omitted:

Proposition: Let d_1, d_2 be equivalent metrics on a set E. Then:

(a) Let $(p_n)_{n=1}^\infty$ be a sequence in E. Then (i) $(p_n)_{n=1}^\infty$ is d_1-convergent if and only if it is d_2-convergent, and (ii) in the convergent case, $\lim^{(d_1)}_{n\to\infty} p_n = \lim^{(d_2)}_{n\to\infty} p_n$.

(b) A sequence $(p_n)_{n=1}^\infty$ in E is d_1-Cauchy if and only if it is d_2-Cauchy.

(c) A sequence $(p_n)_{n=1}^\infty$ in E is d_1-bounded if and only if it is d_2-bounded.

(d) (E, d_1) is complete if and only if (E, d_2) is complete.

Part (a) of this Proposition was proven in class (even though a(ii) was omitted from the *statement* of the Proposition). Prove parts (b), (c), and (d). To avoid labeling-confusion, call these parts (b), (c), and (d) of this problem; treat this problem as having no part (a).

Remark. Parts (a), (b), and (c) of this Proposition, along with with two other facts we proved in class, summarized as saying: *Equivalent metrics determine the same convergent sequences, the same limits of convergent sequences, the same Cauchy sequences, the same bounded sequences, the same open sets, and the same bounded sets.*

B4. Let (E, d) be a metric space, let $\{p_n\}_{n=1}^{\infty}$ be a Cauchy sequence in E, and assume that this sequence has a convergent subsequence $\{p_{n_i}\}_{i=1}^{\infty}$. Let $p = \lim_{i\to\infty} p_{n_i}$. Show that the original sequence $\{p_n\}_{n=1}^{\infty}$ also converges to p.

(Note (E, d) is not assumed to have any properties other than being a metric space; e.g. we are not assuming (E, d) is complete or sequentially compact. The hypotheses say only that *this particular* Cauchy sequence $\{p_n\}_{n=1}^{\infty}$ has a convergent subsequence, not that every sequence has a convergent subsequence, and not that every Cauchy sequence has a convergent subsequence.)

B5. Let (E_1, d_1) and (E_2, d_2) be metric spaces. In earlier homework (Rosenlicht, p. 61/1c) you showed that the function $d : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

is a metric on $E_1 \times E_2$.

(a) Show that the function $d' : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d'((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

is also a metric on $E_1 \times E_2$. (Note: "Show" means "Prove".)

(b) Show that the metrics d and d' on $E_1 \times E_2$ are equivalent.

(c) Show that if (E_1, d_1) and (E_2, d_2) are complete, then so are $(E_1 \times E_2, d)$ and $(E_1 \times E_2, d')$.

B6. Let (E, d) be a sequentially compact metric space. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be an open cover of E, where A is any index-set. Prove that there exists $r > 0$ such that for all $p, q \in E$ with $d(p, q) < r$, there exists $\alpha \in A$ such that p and q both lie in U_α.

Hint. Prove this by contradiction. Start by showing that if the conclusion is false, then there exist sequences $\{p_n\}, \{q_n\}$ in E such that $d(p_n, q_n) < \frac{1}{n}$ but such that there exists no α for which p and q both lie in U_α. (This is a hint as to how to *start* the proof; there's still a fair bit of work to be done after this start.)

Remark. Since "compact" implies "sequentially compact", the result proven in this problem remains true if the word "sequentially" is deleted from the hypotheses.