MAA 4211, Fall 2015—Assignment 6's non-book problems

B1. Let $\mathbf{R}^\infty$ denote the set of sequences in $\mathbf{R}$ (an element $\vec{a} \in \mathbf{R}^\infty$ is an infinite sequence $(a_i)_{i=1}^\infty$ in $\mathbf{R}$). The set $\mathbf{R}^\infty$ is a vector space, with addition of vectors and multiplication by scalars defined "componentwise", just as we do for $\mathbf{R}^n$: if $\vec{a}, \vec{b} \in \mathbf{R}^\infty$ and $c \in \mathbf{R}$, we define

$$\vec{a} + \vec{b} = (a_i + b_i)_{i=1}^\infty, \quad c\vec{a} = (ca_i)_{i=1}^\infty.$$

The zero element $\vec{0}$ of $\mathbf{R}^\infty$ is the sequence each of whose terms is 0.

Let $\mathbf{R}_b^\infty \subset \mathbf{R}^\infty$ denote the subset consisting of all *bounded* sequences. It is not hard to show that $\mathbf{R}_b^\infty$ is a vector subspace of $\mathbf{R}^\infty$; you already did part of the relevant work in doing Rosenlicht problem p. 61/#1b. The ℓ^∞ norm on $\mathbf{R}_b^\infty$ is defined by $\|\vec{a}\|_\infty = \sup\{|a_i| \ : \ i \in \mathbf{N}\}$. In the process of doing Rosenlicht p. 61/#1b, you also did some of the work needed to show that $\| \ \|_\infty$ is a norm on $\mathbf{R}_b^\infty$.

(a) Do all the remaining work needed to show that $\mathbf{R}_b^\infty$ is a vector space (a vector subpace of $\mathbf{R}^\infty$), that $\| \ \|_\infty$ is a norm on $\mathbf{R}_b^\infty$.

The normed vector space $(\mathbf{R}_b^\infty, \| \ \|_\infty)$ is denoted $\ell^\infty(\mathbf{R})$. As with any normed vector space, when we speak of metric-space properties of $\ell^\infty(\mathbf{R})$, the metric is assumed to be the one associated with the given norm (unless otherwise specified); thus the ℓ^∞ metric on $\mathbf{R}_b^\infty$ is the function $d : \mathbf{R}_b^\infty \times \mathbf{R}_b^\infty \to \mathbf{R}$ given by $d(\vec{a}, \vec{b}) = d_\infty(\vec{a}, \vec{b}) = \sup\{|a_i - b_i| \ : \ i \in \mathbf{N}\}$ (exactly the metric in Rosenlicht p. 61/#1b).

Since a sequence in $\ell^\infty(\mathbf{R})$ is a sequence of sequences, to avoid confusion in this problem we will use a superscript rather than a subscript to label the terms of a sequence in $\ell^\infty(\mathbf{R})$; we will write such a sequence as $(\vec{a}^{(n)})_{n=1}^\infty$. Thus the n^{th} term in such a sequence is a real-valued sequence $\vec{a}^{(n)} = (a_i^{(n)})_{i=1}^\infty = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots)$.

(b) Let $(\vec{a}^{(n)})_{n=1}^\infty$ be a Cauchy sequence in $\ell^\infty(\mathbf{R})$. Show that for all $i \in \mathbf{N}$, the real-valued sequence $(a_i^{(n)})_{n=1}^\infty$ (the sequence of "i^{th} components" of the $\vec{a}^{(n)}$) is a Cauchy sequence in $\mathbf{R}$. Note that in $(a_i^{(n)})_{n=1}^\infty$, the index i is fixed; it is n that varies: $(a_i^{(n)})_{n=1}^\infty = (a_i^{(1)}, a_i^{(2)}, a_i^{(3)}, \dots)$.

(c) Let $(\vec{a}^{(n)})_{n=1}^\infty$ be as in part (a). Since $\mathbf{R}$ is complete, for all $i \in \mathbf{N}$ there exists $c_i \in \mathbf{R}$ such that $\lim_{n\to\infty} a_i^{(n)} = c_i$. Let $\vec{c}$ be the sequence $(c_i)_{i=1}^\infty \in \mathbf{R}^\infty$—no subscript "$b$", yet. Show that the sequence $\vec{c}$ *is*, in fact, bounded. (So $(c_i)_{i=1}^\infty \in \mathbf{R}_b^\infty$ after all.)

(d) Let $(\vec{a}^{(n)})_{n=1}^\infty$ and $\vec{c}$ be as in part (b). Show that $(\vec{a}^{(n)})_{n=1}^\infty$ converges in $\ell^\infty(\mathbf{R})$ to $\vec{c}$. (Note: unlike for sequences in $\mathbf{R}^m$, this CANNOT be deduced just from the fact that $(a_i^{(n)})_{n=1}^\infty$ converges to c_i for all i; see part (e) below.) Thus $\ell^\infty(\mathbf{R})$ is complete.

Hint: For $\epsilon > 0$, if $N \in \mathbf{N}$ is as in the Cauchy criterion for the sequence $(\vec{a}^{(n)})_{n=1}^\infty$ in $\ell^\infty(\mathbf{R})$, show that for all $i \in \mathbf{N}$, this same N "works" in the Cauchy criterion for the real-valued sequence $(a_i^{(n)})_{n=1}^\infty$. (You probably already did this in part (b).) Then apply a lemma proved in class on 11/13/15 to each sequence $(a_i^{(n)})_{n=1}^\infty$.

Just FYI: A complete normed vector space is called a *Banach space.*

Notation for the remaining parts of this problem. For $n \in \mathbf{N}$, let $\vec{e}^{(n)} \in \mathbf{R}_b^\infty$ be the sequence whose n^{th} term is 1 and all of whose other terms are zero (e.g. $\vec{e}^{(3)} = (0,0,1,0,0,0,0,\dots)$).

(e) Show that for all $i \in \mathbf{N}$, $(e_i^{(n)})_{n=1}^\infty$ converges in $\mathbf{R}$ to 0.

(f) Let $\vec{0}$ be the zero element of $\mathbf{R}_b^\infty$ (the sequence $(0,0,0,0\dots)$). Compute $d(\vec{e}^{(n)}, \vec{0})$ for all n, and use your answer to show that $(\vec{e}^{(n)})_{n=1}^\infty$ does not converge in $\ell^\infty(\mathbf{R})$ to $\vec{0}$, even though the i^{th}-component sequence $(e_i^{(n)})_{n=1}^\infty$ converges to the i^{th} component of $\vec{0}$ for all i.

(g) Compute $d(\vec{e}^{(n)}, \vec{e}^{(m)})$ for all $m, n \in \mathbf{N}, m \neq n$. Use your answer to show that no subsequence of $(\vec{e}^{(n)})_{n=1}^\infty$ can be Cauchy. Use this to deduce that no subsequence of $(\vec{e}^{(n)})_{n=1}^\infty$ can converge.

Note: since any sequence is trivially a subsequence of itself, the last conclusion implies that $(\vec{e}^{(n)})_{n=1}^\infty$ does not converge in $\ell^\infty(\mathbf{R})$ to *anything*, so, in particular, it does not converge to $\vec{0}$. But I still want you to do part (f) by the method indicated in part (f).)

(h) Use part (g) to deduce that the closed unit ball $\overline{B}_1(\vec{0}) \subset \ell^\infty(\mathbf{R})$ is not sequentially compact.

> **Remark**. Thus, by parts (d) and (h),$\overline{B}_1(\vec{0}) \subset \ell^\infty(\mathbf{R})$ is a closed, bounded subset of a complete normed vector space, yet is not sequentially compact, hence is not compact (or totally bounded). The Heine-Borel Theorem (one version of which asserts that closed, bounded subsets of $\mathbf{R}^m$, with respect to the metric given by any norm, are compact), which we have proven for any norm on $\mathbf{R}^m$ equivalent to the ℓ^∞ norm, does not extend to infinite-dimensional vector spaces.

Problems B2 and B3 are intended to help you better relate the definition of "connected subset of a metric space" to the intuitive notion of what it sounds like this terminology ought to mean.

B2. Let (E, d) be a metric space, $S \subset E$. Prove that the following are equivalent:

(i) S is not connected.

(ii) There exist nonempty subsets $A, B \subset S$ for which $S = A \bigcup B$ and for which $\overline{A} \bigcap B = \emptyset = A \bigcap \overline{B}$. (Here $\overline{A}$ and $\overline{B}$ denote the closures of A and B in E, not in the subspace S.)

Some things to note: (1) Under the conditions on sets A, B in (ii), we automatically have $A \bigcap B = \emptyset$, so $S = A \coprod B$. For arbitrary subsets $A, B \subset E$, the condition "$\overline{A} \bigcap B = \emptyset$" is *stronger* (more restrictive) than "$A \bigcap B = \emptyset$." (2) In (ii), we are *not* assuming that A and B are open in S. (Openness will end up being a *consequence* of what we've assumed, but it's not one of our assumptions.) (3) The equivalence of (i) and (ii) is interesting only for *proper* subsets $S \subsetneq E$. When $S = E$, the equivalence follows immediately from the definition of "connected metric space".

B3. Let (E, d) be a metric space, $S \subset E$ a nonempty subset, and $p \in E$. The *distance from p to S*, which we will write as $\text{dist}(p, S)$, is defined to be $\inf\{d(p,q) \mid q \in S\}$.

(a) Prove that $\text{dist}(p, S) = 0$ if and only if $p \in \overline{S}$. You may use any facts stated in the Interiors, Closures, and Boundaries handout.

(b) Using part (a) and the result of B2, prove that the following are equivalent:

(i) S is not connected.

(ii) $S = A \bigcup B$ for some nonempty sets $A, B \subset S$ for which every point of each set is a positive distance from the other set (i.e. $\text{dist}(p, B) > 0 \ \forall p \in A$ and $\text{dist}(p, A) > 0 \ \forall p \in B$).

Motivation for the above problem: Recall that, heuristically, we wanted "S is not connected" to mean that S cannot be partitioned into two nonempty disjoint subsets that "don't touch each other". There is no official definition of one subset of a metric space *touching*, or not touching, another. However, were we (not unreasonably) to define "A does not touch B" to mean "every point of A is a positive distance from B", then the characterization of non-connectedness in this problem would turn the heuristic characterization of "not connected" into a precise one that agrees with the mathematical definition.

B4. Let (E, d) be a metric space, $S \subset E$ a connected subset. Prove that the closure of S is connected.