MAA 4211, Fall 2015—Assignment 7's non-book problems

B1. Let X, Y be nonempty sets, let d_X, d'_X be equivalent metrics on X, and let d_Y, d'_Y be equivalent metrics on Y. Prove that if a function $f : X \to Y$ is continuous as a map $(X, d_X) \to (Y, d_Y)$, then f is continuous as a map $(X, d'_X) \to (Y, d'_Y)$. (In particular this holds if we vary only one of the metrics, i.e. if $d_X = d'_X$ or $d_Y = d'_Y$.)

B2. In Rosenlicht p. 91/#3, suppose you remove the hypothesis that the sets S_1, S_2 are both closed. Is the conclusion still true? (Prove your answer, of course.)

B3. Use the result of Rosenlicht p. 91/#2 to turn the previously difficult problem p. 61/#6 into a nearly trivial problem. (I.e. solve the same problem very easily, using the tools we now have.)

B4. (This problem gives a second proof of a lemma we proved in class.) Let (E, d) be a metric space, let $\{p_n\}_{n=1}^{\infty}$ be a convergent sequence in E, let $p = \lim_{n\to\infty} p_n$, let $N \in \mathbf{N}$, and let $\epsilon > 0$. Assume that for all $n, m \geq N$, $d(p_n, p_m) < \epsilon$. Use the appropriate half of the "sequential characterization of continuity" to prove that $d(p_N, p) \leq \epsilon$. *Hint*: Consider the function $f : E \to \mathbf{R}$ defined by $f(q) = d(p_N, q)$.

In problems B5 below, to do any problem-part you are allowed to assume the result(s) of *earlier* parts of the same problem. Each part is essentially a hint for the next part, with the exception that the first three parts of B5 will not help you with B5(d).

B5. In this problem you will prove the following important theorem:

Theorem 1: ***For any $n \geq 0$, all norms on $\mathbf{R}^n$ are equivalent (i.e. any two norms are equivalent to each other).***

For $n = 0$, the assertion in the theorem is true trivially ($\mathbf{R}^0 = \{0\}$, and the only norm is the zero-function), so we will focus on $n \geq 1$. For the rest of this problem, let $n \geq 1$ be fixed, let $\{e_i\}_{i=1}^{n}$ denote the standard basis of $\mathbf{R}^n$, let $\| \ \|_\infty$ denote the ℓ^∞ norm on $\mathbf{R}^n$, and let d_∞ denote the associated metric on $\mathbf{R}^n$. For parts (a)–(h), let $\| \ \|$ denote a fixed (but arbitrary) norm on $\mathbf{R}^n$, and let d denote the metric on $\mathbf{R}^n$ associated with the norm $\| \ \|$. Each part of the problem can be done in a few lines—in some cases, just one or two lines—so don't be worried by the number of parts; the parts are numerous to *help* you prove Theorem 1. (You'd find the problem much more difficult if parts (a)-(h) were omitted and the problem said only "Prove Theorem 1.")

(a) Show that there exists $M > 0$ such that

$$\|v\| \leq M\|v\|_\infty \quad \text{for all } v \in \mathbf{R}^n \ . \tag{1}$$

Hint: observe that "$v = (v_1, v_2, \ldots, v_n)$" can be rewritten as "$v = \sum_i v_i e_i$", and use the (iterated) triangle inequality.

(b) Define $f : \mathbf{R}^n \to \mathbf{R}$ by $f(v) = \|v\|$. Show that for all $v, w \in \mathbf{R}^n$,

$$|f(v) - f(w)| \leq M\|v - w\|_\infty, \tag{2}$$

where M is the same as in inequality (1).

(c) Show that $f : (\mathbf{R}^n, d_\infty) \to \mathbf{R}$ is continuous.

Note: $\|v\| = d(v, 0)$, and we proved in class that for any metric space (E', d') and $q \in E$, the function $p \mapsto d'(p, q)$ is continuous *with respect to the metric* d. But here, you are showing that the function $v \mapsto d(v, 0)$ is continuous *with respect to the metric* d_∞, a different metric. Thus the continuity of f is not an immediate corollary of anything we proved in class.

(d) Let $S = \{v \in \mathbf{R}^n \mid \|v\|_\infty = 1\}$ (the unit sphere centered at the origin in $(\mathbf{R}^n, d_\infty)$). Show that S is closed and bounded in $(\mathbf{R}^n, d_\infty)$. Conclude from the Heine-Borel Theorem that S is a compact subset of $(\mathbf{R}^n, d_\infty)$.[1]

(e) Let $g = f|_S$ (the restriction of f to S). Show that g achieves a minimum value m, and show that $m > 0$.

(f) Show that for all nonzero $v \in \mathbf{R}^n$,

$$\left\| \frac{v}{\|v\|_\infty} \right\| \geq m, \tag{3}$$

where m is as in part (e).

(g) Show that for all $v \in \mathbf{R}^n$ (including the zero vector),

$$\|v\| \geq m\|v\|_\infty \,. \tag{4}$$

(h) Show that the norm $\| \; \|$ is equivalent to the norm $\| \; \|_\infty$.

(i) Prove Theorem 1.

[1] In class we proved the Heine-Borel Theorem for $(\mathbf{R}^n, d_\infty)$, and deduced that it holds for $(\mathbf{R}^n, d')$, where d' is any metric equivalent to d_∞. But so far the only metrics we've shown to be equivalent to d_∞ are the metrics associated with the ℓ^1, ℓ^2, and ℓ^∞ norms. Thus these are the only three metrics on $\mathbf{R}^n$ for which we know, so far, that closed-and-bounded implies compact. Once Theorem 1 is proven, we will know that closed-and-bounded implies compact for *any* metric on $\mathbf{R}^n$ that comes from a norm.