

MAA 4211, Fall 2016—Assignment 3’s non-book problems

B1. (a) Let Z be any nonempty set, and let $\text{Fun}(Z, \mathbf{R})$ denote the set of all functions $Z \rightarrow \mathbf{R}$. Temporary notation, just for this problem: let $\mathbf{0}$ denote the constant function with value 0 (i.e. $\mathbf{0}(z) = 0$ for all $z \in Z$). For $f, g \in \text{Fun}(Z, \mathbf{R})$ and $c \in \mathbf{R}$ we define $f + g \in \text{Fun}(Z, \mathbf{R})$ and $cf \in \text{Fun}(Z, \mathbf{R})$ by

$$\begin{aligned} f + g &= \text{the function } z \mapsto f(z) + g(z), \\ cf &= \text{the function } z \mapsto cf(z). \end{aligned}$$

Check that, with the operations above, $\text{Fun}(Z, \mathbf{R})$ is a vector space with zero-element $\mathbf{0}$.

(b) Let \mathbf{R}^∞ denote the set of all functions $\mathbf{N} \rightarrow \mathbf{R}$. For $f \in \mathbf{R}^\infty$, one of the notations we commonly use is (x_1, x_2, x_3, \dots) , where $x_n = f(n), n \in \mathbf{N}$. Thus an element of \mathbf{R}^∞ is also called an *infinite sequence* in \mathbf{R} . By part (a), \mathbf{R}^∞ is a vector space. Check that, in the sequence-notation above, the operations and zero-element in \mathbf{R}^∞ (as defined in part (a)) are given by

$$\begin{aligned} (x_1, x_2, x_3, \dots) + (y_1, y_2, y_3, \dots) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \dots), \\ c(x_1, x_2, x_3, \dots) &= (cx_1, cx_2, cx_3, \dots), \\ \mathbf{0} &= (0, 0, 0, \dots). \end{aligned}$$

(c) A real-valued sequence $\vec{x} = (x_1, x_2, x_3, \dots) \in \mathbf{R}^\infty$ (switching from the notation “ f ” in part (b)) is called *bounded* if there exists $M \in \mathbf{R}$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$; equivalently, if the set $\{|x_n| : n \in \mathbf{N}\}$ is bounded above. Let $\mathbf{R}_b^\infty \subset \mathbf{R}^\infty$ denote the set of bounded real-valued sequences. Show that \mathbf{R}_b^∞ is a vector subspace of \mathbf{R}^∞ .

(d) For any $\vec{x} \in \mathbf{R}_b^\infty$, the set $\{|x_n| : n \in \mathbf{N}\}$ is nonempty and bounded above, hence has a least upper bound. Therefore we can define a function $\|\cdot\|_\infty : \mathbf{R}_b^\infty \rightarrow \mathbf{R}$ by

$$\|\vec{x}\|_\infty := \sup\{|x_n| : n \in \mathbf{N}\}.$$

Show that $\|\cdot\|_\infty$ is a norm on the vector space \mathbf{R}_b^∞ . (Note: the “ ∞ ” subscript in $\|\cdot\|_\infty$ has nothing to do with the “ ∞ ” superscript in \mathbf{R}^∞ . Rather, the notation for this norm comes from the analogous norm on \mathbf{R}^n , where we can replace “sup” by “max”.) We call this norm the ℓ^∞ -norm or *sup-norm* on \mathbf{R}_b^∞ .

(e) Let d_∞ denote the metric on \mathbf{R}_b^∞ associated with the ℓ^∞ norm; we call d_∞ the ℓ^∞ metric on \mathbf{R}_b^∞ . Check that d_∞ is given by

$$d_\infty(\vec{x}, \vec{y}) = \sup\{|x_n - y_n| : n \in \mathbf{N}\}.$$

B2. Let $(V, \|\cdot\|)$ be a normed vector space, viewed as a metric space with the associated metric. Show that for all $v \in V$,

for each $r > 0$ we have $B_r(v) = \{v + w \mid w \in B_r(0)\}$,
and for each $r \geq 0$ we have $\overline{B}_r(v) = \{v + w \mid w \in \overline{B}_r(0)\}$.

In other words, each open (respectively, closed) ball centered at a given v is simply the translation, by v , of the open (respectively, closed) ball of the same radius centered at the origin.

B3. Define a metric d on the set of rational numbers \mathbf{Q} by $d(x, y) = |x - y|$ (the restriction to \mathbf{Q} of the standard metric on \mathbf{R}). Give an example, with proof, of a nonempty, proper subset of (\mathbf{Q}, d) that is both open and closed in this metric space. (Do not expect your subset to be either open or closed in \mathbf{R} , let alone *both* open and closed in \mathbf{R} . There is no nonempty, proper subset of \mathbf{R} that is both open and closed with respect to the standard metric.)

B4. Let $n \geq 1$ and let \mathbf{E}^n denote Euclidean n -space. Let $p \in \mathbf{E}^n$, $r \geq 0$. Let $\overline{B}_r(p)$ denote the closed ball of radius r centered at p . Prove that $\overline{B}_r(p)$ is not an open set.

Remember: (i) “Closed” does not imply “not open”. The fact that a closed ball in a metric space is a closed set does not imply that a closed ball can’t be an open set. (In fact, in one of the Rosenlicht problems you will see an example in which *every* ball is simultaneously an open set and a closed set.) (ii) There is no such thing as “proof by picture”. If you are asserting, for example, that a certain open ball contains points of some other set, you have to *prove* that assertion, not merely assert that it’s true based on some picture you’ve drawn and your intuition.

B5. Let (E, d) be a metric space. For purposes of this problem, for each $p \in E$ define a property we’ll call “boundedness with respect to p ” as follows: a set $S \subset E$ is *bounded with respect to p* if S is contained in some ball *centered at p* . Show that the following are equivalent, for every $S \subset E$:

- (i) S is bounded with respect to p .
- (ii) S is bounded.
- (iii) S is bounded with respect to q for all $q \in E$.