MAA 4211, Fall 2016—Assignment 5's non-book problems

B1. Let (E, d) be a metric space, let $(p_n)_{n=1}^{\infty}$ be a Cauchy sequence in E, and assume that this sequence has a convergent subsequence $(p_{n_i})_{i=1}^{\infty}$. Let $p = \lim_{i\to\infty} p_{n_i}$. Show that the original sequence $(p_n)_{n=1}^{\infty}$ also converges to p.

(Note (E, d) is not assumed to have any properties other than being a metric space; e.g. we are not assuming (E, d) is complete or sequentially compact. The hypotheses say only that *this particular* Cauchy sequence $(p_n)_{n=1}^{\infty}$ has a convergent subsequence, not that every sequence has a convergent subsequence, and not that every Cauchy sequence has a convergent subsequence.)

B2. Let (E_1, d_1) and (E_2, d_2) be metric spaces. In earlier homework (Rosenlicht, p. 61/1c) you showed that the function $d : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

is a metric on $E_1 \times E_2$. On your first midterm you showed that the function $d' : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d'((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

is also a metric on $E_1 \times E_2$. Show that the metrics d and d' on $E_1 \times E_2$ are equivalent. (Note: "Show" always means "Prove".)

(c) Show that if (E_1, d_1) and (E_2, d_2) are complete, then so are $(E_1 \times E_2, d)$ and $(E_1 \times E_2, d')$.

B3. In previous homework you showed that $\mathbf{R}_b^{\infty}$, the space of bounded real-valued sequences, is a vector subspace of $\mathbf{R}^{\infty}$, the space of all real-valued sequences. Now define

$$\mathbf{R}_0^{\infty} = \{\vec{a} = (a_m)_{m=1}^{\infty} \in \mathbf{R}^{\infty} : a_m = 0 \text{ for all but finitely many } m\}.$$

(a) Show that $\mathbf{R}_0^{\infty}$ is a vector subspace of $\mathbf{R}_b^{\infty}$.

(b) In view of part (a), we can restrict the ℓ^{∞} norm on $\mathbf{R}_b^{\infty}$, as well as its associated metric d_{∞}, to $\mathbf{R}_0^{\infty}$. (Note that for $\vec{a} \in \mathbf{R}_0^{\infty}$, we can actually write $\|\vec{a}\|_{\infty} = \max\{|a_m| : m \in \mathbf{N}\}$ instead of $\|\vec{a}\|_{\infty} = \sup\{|a_m| : m \in \mathbf{N}\}$; "max" is well-defined here, since for each $\vec{a} \in \mathbf{R}_0^{\infty}$, the set $\{|a_m| : m \in \mathbf{N}\}$ is finite.) Show that $(\mathbf{R}_0^{\infty}, d_{\infty})$ is not complete. I.e. produce a sequence $(\vec{a}^{(i)})_{i=1}^{\infty}$ in $\mathbf{R}_0^{\infty}$ that is Cauchy but does not converge in $(\mathbf{R}_0^{\infty}, d_{\infty})$. (Hint: consider sequences $(\vec{a}^{(i)})_{i=1}^{\infty}$ in which $a_j^{(i)} = 0$ for $j > i$ but $a_i^{(i)} \neq 0$, and for which the only difference between $\vec{a}^{(i)}$ and $\vec{a}^{(i-1)}$ is the i^{th} term.)

B4. (Strengthening of Rosenlicht problem III.10.) Let (p_n) be a convergent sequence in a metric space (E, d) and let $p = \lim_{n\to\infty} p_n$. Show that $\text{range}(p_n) \bigcup \{p\}$ is a compact subset of E.

B5. Let (E, d) be a metric space and let $S \subset E$.

(a) Prove that $\overline{S} = S \bigcup \{\text{all cluster points of } S\}$.

(b) Prove that $\overline{S} = S \bigcup \{\text{all cluster points of } S \text{ that lie in } \partial S\}$.

B6. Let d_1, d_2 be equivalent metrics on a set E. Without using any relations between compactness and sequential compactness (none of which we've discussed as of the date this problem is being posted), prove that (E, d_1) is sequentially compact if and only if (E, d_2) is sequentially compact.

B7. Notation as in B3, but here we will be interested in the whole space $\mathbf{R}_b^\infty$, not the subspace $\mathbf{R}_0^\infty$.

The normed vector space $(\mathbf{R}_b^\infty, \| \ \|_\infty)$ is conventionally called $\ell^\infty(\mathbf{R})$. As with any normed vector space, when we speak of metric-space properties of $\ell^\infty(\mathbf{R})$, the metric is assumed to be the one associated with the given norm (unless otherwise specified); thus the ℓ^∞ metric on $\mathbf{R}_b^\infty$ is the function $d : \mathbf{R}_b^\infty \times \mathbf{R}_b^\infty \to \mathbf{R}$ given by $d(\vec{a}, \vec{b}) = d_\infty(\vec{a}, \vec{b}) = \sup\{|a_i - b_i| \ : \ i \in \mathbf{N}\}$ (exactly the metric in Rosenlicht p. 61/#1b).

Since a sequence in $\ell^\infty(\mathbf{R})$ is a sequence of sequences, to avoid confusion in this problem we will use a superscript rather than a subscript to label the terms of a sequence in $\ell^\infty(\mathbf{R})$; we will write such a sequence as $(\vec{a}^{(n)})_{n=1}^\infty$. Thus the n^{th} term in such a sequence is a real-valued sequence $\vec{a}^{(n)} = (a_i^{(n)})_{i=1}^\infty = (a_1^{(n)}, a_2^{(n)}, a_3^{(n)}, \dots)$.

(a) Let $(\vec{a}^{(n)})_{n=1}^\infty$ be a Cauchy sequence in $\ell^\infty(\mathbf{R})$. Show that for all $i \in \mathbf{N}$, the real-valued sequence $(a_i^{(n)})_{n=1}^\infty$ (the sequence of "i^{th} components" of the $\vec{a}^{(n)}$) is a Cauchy sequence in $\mathbf{R}$. Note that in $(a_i^{(n)})_{n=1}^\infty$, the index i is fixed; it is n that varies: $(a_i^{(n)})_{n=1}^\infty = (a_i^{(1)}, a_i^{(2)}, a_i^{(3)}, \dots)$.

(b) Let $(\vec{a}^{(n)})_{n=1}^\infty$ be as in part (a). Since $\mathbf{R}$ is complete, for all $i \in \mathbf{N}$ there exists $c_i \in \mathbf{R}$ such that $\lim_{n\to\infty} a_i^{(n)} = c_i$. Let $\vec{c}$ be the sequence $(c_i)_{i=1}^\infty \in \mathbf{R}^\infty$—no subscript "$b$", yet. Show that the sequence $\vec{c}$ *is*, in fact, bounded. (So $(c_i)_{i=1}^\infty \in \mathbf{R}_b^\infty$ after all.)

(c) Let $(\vec{a}^{(n)})_{n=1}^\infty$ and $\vec{c}$ be as in part (b). Show that $(\vec{a}^{(n)})_{n=1}^\infty$ converges in $\ell^\infty(\mathbf{R})$ to $\vec{c}$. (Note: unlike for sequences in $\mathbf{R}^m$, this CANNOT be deduced just from the fact that $(a_i^{(n)})_{n=1}^\infty$ converges to c_i for all i; see part (e) below.) Thus $\ell^\infty(\mathbf{R})$ is complete.

Hint: For $\epsilon > 0$, if $N \in \mathbf{N}$ is as in the Cauchy criterion for the sequence $(\vec{a}^{(n)})_{n=1}^\infty$ in $\ell^\infty(\mathbf{R})$, show that for all $i \in \mathbf{N}$, this same N "works" in the Cauchy criterion for the real-valued sequence $(a_i^{(n)})_{n=1}^\infty$. (You probably already did this in part (b).) Then apply a lemma proved in class on 11/7/16 to each sequence $(a_i^{(n)})_{n=1}^\infty$.

Notation for the remaining parts of this problem. For $n \in \mathbf{N}$, let $\vec{e}^{(n)} \in \mathbf{R}_b^\infty$ be the sequence whose n^{th} term is 1 and all of whose other terms are zero (e.g. $\vec{e}^{(3)} = (0, 0, 1, 0, 0, 0, 0, \dots)$).

(d) Show that for all $i \in \mathbf{N}$, $(e_i^{(n)})_{n=1}^{\infty}$ converges in $\mathbf{R}$ to 0.

(e) Let $\vec{0}$ be the zero element of $\mathbf{R}_b^{\infty}$ (the sequence $(0, 0, 0, 0 \ldots)$). Compute $d(\vec{e}^{(n)}, \vec{0})$ for all n, and use your answer to show that $(\vec{e}^{(n)})_{n=1}^{\infty}$ does not converge in $\ell^{\infty}(\mathbf{R})$ to $\vec{0}$, even though the i^{th}-component sequence $(e_i^{(n)})_{n=1}^{\infty}$ converges to the i^{th} component of $\vec{0}$ for all i.

(f) Compute $d(\vec{e}^{(n)}, \vec{e}^{(m)})$ for all $m, n \in \mathbf{N}, m \neq n$. Use your answer to show that no subsequence of $(\vec{e}^{(n)})_{n=1}^{\infty}$ can be Cauchy. Use this to deduce that no subsequence of $(\vec{e}^{(n)})_{n=1}^{\infty}$ can converge.

Note: since any sequence is trivially a subsequence of itself, the last conclusion implies that $(\vec{e}^{(n)})_{n=1}^{\infty}$ does not converge in $\ell^{\infty}(\mathbf{R})$ to *anything*, so, in particular, it does not converge to $\vec{0}$. But I still want you to do part (f) by the method indicated in part (f).)

(g) Use part (f) to deduce that the closed unit ball $\overline{B}_1(\vec{0}) \subset \ell^{\infty}(\mathbf{R})$ is not sequentially compact.