MAA 4211, Fall 2016—Assignment 6's non-book problems

Problems B1 and B2(c) are intended to help you better relate the formal meaning of "connected subset of a metric space" to the intuitive notion of what it sounds like this terminology ought to mean.

B1. Let (E, d) be a metric space, $S \subset E$. Prove that the following are equivalent:

(i) S is not connected.

(ii) There exist nonempty subsets $A, B \subset S$ such that $S = A \bigcup B$ and $\overline{A} \bigcap B = \emptyset = A \bigcap \overline{B}$. (Here $\overline{A}$ and $\overline{B}$ denote the closures of A and B in E, not in the subspace (S, d).)

Some things to note: (1) Under the conditions on sets A, B in (ii), we automatically have $A \bigcap B = \emptyset$, so $S = A \coprod B$. For arbitrary subsets $A, B \subset E$, the condition "$\overline{A} \bigcap B = \emptyset$" is *stronger* (more restrictive) than "$A \bigcap B = \emptyset$." (2) In (ii), we are *not* assuming that A and B are open in (S, d). (Openness in (S, d) will end up being a *consequence* of what we've assumed, but it's not one of our assumptions.) (3) The equivalence of (i) and (ii) is interesting only for *proper* subsets $S \subsetneq E$. When $S = E$, the equivalence follows immediately from the definition of "connected metric space".

B2. Let (E, d) be a metric space, $S \subset E$ a nonempty subset, and $p \in E$. The *distance from p to S*, which we will write as $\text{dist}(p, S)$, is defined to be $\inf\{d(p, q) \mid q \in S\}$.

(a) Prove that $\text{dist}(p, S) = 0$ if and only if $p \in \overline{S}$. You may use any facts stated in the Interiors, Closures, and Boundaries handout.

(b) For $(E, d) = \mathbf{E}^2$, give an example of each of the following.

(i) A subset S and a point $p \notin S$ for which the infimum defining $\text{dist}(p, S)$ *is not* achieved.

(ii) A subset S and a point $p \notin S$ for which the infimum defining $\text{dist}(p, S)$ *is* achieved. (Note that "The infimum defining $\text{dist}(p, S)$ is achieved" is equivalent to "There is a point $q \in S$ that, among all points in S, minimizes distance to p.")

(c) For purposes of this problem, call S *non-connected* if S is not connected.[1] Using part (a) and the result of B1, prove that S is non-connected if and only if $S = A \bigcup B$

[1] Although it is tempting to use the term "disconnected" for "not connected", topologists generally don't do this, instead reserving "disconnected" as one piece of the terminology for topological (sub)spaces that fail in some spectacular way to be connected, such as *totally disconnected* spaces (see problem B4(g)). The most common terminology for "not connected" is "not connected", not "non-connected". Here I am using the term "non-connected" because a sentence of the form "S is not connected if and only if ..." is ambiguous—does it mean that "not connected" is equivalent to the specified conditions, or does it mean "The statement 'S is connected if and only if ...' " is false? Make sure you avoid this sort of ambiguous phrasing in your answers.

for some nonempty sets $A, B \subset S$ for which every point of each set is a positive distance from the other set (i.e. $\mathrm{dist}(p, B) > 0\ \forall p \in A$ and $\mathrm{dist}(p, A) > 0\ \forall p \in B$).

Motivation for part (c): Recall that, heuristically, we wanted "S is not connected" to mean that S cannot be partitioned into two nonempty disjoint subsets that "don't touch each other". There is no official definition of one subset of a metric space *touching*, or not touching, another. However, were we (not unreasonably) to define "A does not touch B" to mean "every point of A is a positive distance from B", then the characterization of non-connectedness in this problem would turn the heuristic characterization of "not connected" into a precise one that agrees with the mathematical definition.

B3. Let (E, d) be a metric space, $S \subset E$ a connected subset. Prove that the closure of S is connected.

B4. Let (E, d) be a metric space.

(a) Let $p \in E$. Show that the singleton set $\{p\}$ is connected.

(b) Let $p \in E$, and let $\mathcal{F}_p = \{S \subset E \mid S \text{ is connected and } p \in S\} \subset P(E)$. Let

$$C_p = \bigcup_{S \in \mathcal{F}_p} S.$$

Prove that C_p is connected.

(Do not re-invent the wheel to prove this. You should need no more than a couple of sentences, if you apply a relevant proposition in Section III.6 of Rosenlicht.)

The set C_p defined above is called the *connected component of p in E* (or in (E, d)). We will use the notation "C_p" with this meaning for the rest of this problem. A subset $C \subset E$ is called a *connected component of E* if $C = C_p$ for some $p \in E$.

(c) For $p \in E$, prove that C_p is the largest connected set containing p, in the following sense: if $S \subset E$ is connected and $p \in S$, then $S \subset C_p$.

(d) Define a relation $\sim$ on E by declaring $p \sim q$ if and only if $q \in C_p$. Prove that $\sim$ is an equivalence relation, and that the equivalence classes are exactly the connected components of E.

Recall that, for any equivalence relation on a set S, the equivalence classes partition S into pairwise disjoint subsets. (For the relation above, "pairwise disjointness" means that for any $p, q \in E$, either $C_p = C_q$ or $C_p \bigcap C_q = \emptyset$.) Thus a metric space is always the disjoint union of its connected components.

(e) Prove that every connected component of (E, d) is a closed subset of E. (Here (E, d) is a general metric space again, not a totally disconnected metric space.)

(f) Use part (e) to prove that if (E, d) has only finitely many connected components, then each connected component is both open and closed.

(g) (E, d) is called *totally disconnected* if the only nonempty connected subsets of E are the singleton sets. Prove that $\mathbf{Q}$, with its usual metric, is totally disconnected.

Note: in Assignment 3, Problem B3, you effectively were proving that $\mathbf{Q}$ is not connected (but "connected" was not in our mathematical vocabulary at the time). Now you are proving something much stronger.

B5. Let X, Y be nonempty sets, let d_X, d'_X be equivalent metrics on X, and let d_Y, d'_Y be equivalent metrics on Y. Prove that if a function $f : X \to Y$ is continuous as a map $(X, d_X) \to (Y, d_Y)$, then f is continuous as a map $(X, d'_X) \to (Y, d'_Y)$. (In particular this holds if we vary only one of the metrics, i.e. if $d_X = d'_X$ or $d_Y = d'_Y$.)

B6. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : (X, d_X) \to (Y, d_Y)$ a continuous function, and $S \subset X$ nonempty. Prove that the restriction $f|_S$ is continuous as a function $(S, d_X) \to (Y, d_Y)$.