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The Extended Reals

The "extended real number system" ("extended reals", for short), denoted $\mathbf{R}_{\text{ext}}$ in these notes, is a convenient way of dealing with subsets of $\mathbf{R}$ and sequences in $\mathbf{R}$ that are (potentially) unbounded. Before we define $\mathbf{R}_{\text{ext}}$, however, it is important to state what $\mathbf{R}_{\text{ext}}$ is *not*: $\mathbf{R}_{\text{ext}}$ is *not* a field, and $\mathbf{R}_{\text{ext}}$ is *not* a metric space. The extended reals share many properties with the reals, *but not all.* You should take care not to make any implicit assumptions about $\mathbf{R}_{\text{ext}}$ that the notation and terminology may tempt you to make.

1 The ordered set $\mathbf{R}_{\text{ext}}$

Let $(\mathbf{F}, <)$ be an ordered field. Define a set $\mathbf{F}_{\text{ext}}$ to be the disjoint union of $\mathbf{F}$ and two other elements, which we will call Arnold and Zelda:

$$\mathbf{F}_{\text{ext}} = \mathbf{F} \coprod \{\text{Arnold}, \text{Zelda}\}.$$

We extend the relation "$<$" from $\mathbf{F}$ to $\mathbf{F}_{\text{ext}}$ by declaring Arnold $< x$ and $x <$ Zelda for all $x \in \mathbf{F}$, and Arnold $<$ Zelda.[1] (We do not extend "$<$" any further; i.e. there are no other ordered pairs $(a, b) \in \mathbf{F}_{\text{ext}} \times \mathbf{F}_{\text{ext}}$, with at least one of a, b in the set $\{\text{Arnold}, \text{Zelda}\}$, such that $a < b$.)

Exercises

1. Show that the relation $<$ on $\mathbf{F}_{\text{ext}}$ is transitive.
2. Show that the pair $(\mathbf{F}_{\text{ext}}, <)$ obeys trichotomy; i.e. show that for all $x, y \in \mathbf{F}_{\text{ext}}$, exactly one of the following is true: $x < y$, $y < x$, or $x = y$.

Remark. A *totally ordered* or *linearly ordered* set is a pair $(S, <)$, where S is a set and $<$ is a transitive relation on S for which trichotomy holds. Thus if an ordered field $(\mathbf{F}, <)$ is a totally ordered set, and so is $(\mathbf{F}_{\text{ext}}, <)$.

From "$<$", we define relations "$\leq$, "$>$", and "$\geq$" on $\mathbf{F}_{\text{ext}}$ just as we did for ordered fields ("$x \leq y$" means "$x < y$ or $x = y$"; "$x > y$" means "$y < x$", etc.). Using these extended relations, we define the notions of *upper bound*, *lower bound*, *least upper bound*, and *greatest lower bound* of subsets of $\mathbf{F}_{\text{ext}}$ just as we did for the ordered field

[1] Note that "Arnold $<$ Zelda" is part of the *definition* of the extended relation; it does not follow from "Arnold $< x$ and $x <$ Zelda for all $x \in \mathbf{F}$." We proved earlier this semester that the relation "$<$" *on an ordered field* $\mathbf{F}$ is transitive, but now we are using the same symbol "$<$" for a relation on a different set. The symbol "$<$" would be a perverse choice of notation if we were defining a relation that is *not* transitive, but that fact doesn't *prove* that "$<$" is transitive on $\mathbf{F}_{\text{ext}}$. There is no such thing as "proof by notation".

$\mathbf{F}$. Note that *every* subset of $\mathbf{F}_{\text{ext}}$ is bounded from above by Zelda, and bounded from below by Arnold.

Exercise

3. Let us say that $\mathbf{F}_{\text{ext}}$ *has Property B* if every subset of $\mathbf{F}_{\text{ext}}$ has a least upper bound (in $\mathbf{F}_{\text{ext}}$). Show that $\mathbf{F}$ has the Least Upper Bound Property if and only if $\mathbf{F}_{\text{ext}}$ has Property B. (Note: "Property B" is terminology invented purely for these notes; it is not a standard term.)

Henceforth the only ordered field we consider is $\mathbf{R}$, the unique ordered field with the Least Upper Bound Property. We also henceforth use the notation "∞" for Zelda, and "$-\infty$" for Arnold. We also allow the notation "$+\infty$" for "∞", so that the notation "$\pm\infty$" makes sense.

Exercise

4. Show that every subset of $\mathbf{R}_{\text{ext}}$ has a greatest lower bound (in $\mathbf{R}_{\text{ext}}$).

Note that by Exercises 3 and 4, every subset of $\mathbf{R}_{\text{ext}}$ has a least upper bound and greatest lower bound $\underline{\text{in } \mathbf{R}_{\text{ext}}}$. However, to maintain the distinction between subsets of $\mathbf{R}$ that are bounded from above $\underline{\text{in } \mathbf{R}}$ and those that are not, *we do not use the notation "l.u.b."* when we are talking about subsets of $\mathbf{R}$ that are (potentially) unbounded from above. Similarly, we do not use the notation "g.l.b." when talking about subsets of $\mathbf{R}$ that are (potentially) unbounded from below. Instead, we have the following notation and terminology (both for subsets of $\mathbf{R}$ and, more generally for subsets of $\mathbf{R}_{\text{ext}}$):

Definition 1.1 Let A be a nonempty subset of $\mathbf{R}_{\text{ext}}$. The *supremum* of A, denoted $\sup(A)$, is the least upper bound $\underline{\text{in } \mathbf{R}_{\text{ext}}}$ of A. The *infimum* of A, denoted $\inf(A)$, is the greatest lower bound $\underline{\text{in } \mathbf{R}_{\text{ext}}}$ of A.

Observe that if A is a bounded, nonempty subset of $\mathbf{R}$, then the above definition of $\sup(A)$ and $\inf(A)$ coincides with the definition we have been using this semester. Thus, Definition 1.1 merely *extends* our previous definition to cover more cases.

Also observe that if $A \subset \mathbf{R}_{\text{ext}}$ is nonempty, and $x \in A$, then $\inf(A) \leq x \leq \sup(A)$. Thus, by transitivity,

$$\inf(A) \leq \sup(A). \tag{1.1}$$

The reason for requiring A to be nonempty in Definition 1.1 is to guarantee that inequality (1.1) *is satisfied for all sets A for which* $\inf(A)$ *and* $\sup(A)$ *are defined.* Every element of $\mathbf{R}_{\text{ext}}$ is an upper bound and a lower bound of the empty set $\emptyset$, so that the least upper bound of $\emptyset$ in $\mathbf{R}_{\text{ext}}$ is $-\infty$, while the greatest lower bound of $\emptyset$ in $\mathbf{R}_{\text{ext}}$ is ∞. Were we to allow $A = \emptyset$ in Definition (1.1), the inequality (1.1) would fail spectacularly; we would have $\inf(\emptyset) = \infty$ and $\sup(\emptyset) = -\infty$.

2 Sequences and order

Definition 2.1 Let $\{a_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathbf{R}_{\text{ext}}$. We say that $\{a_n\}$ *converges (in* $\mathbf{R}_{\text{ext}}$*) to* ∞ if for all $c \in \mathbf{R}$, there exists $N \in \mathbf{N}$ such that $a_n > c$ for all $n \geq N$. Similarly, we say that $\{a_n\}$ *converges (in* $\mathbf{R}_{\text{ext}}$*) to* $-\infty$ if for all $c \in \mathbf{R}$, there exists $N \in \mathbf{N}$ such that $a_n < c$ for all $n \geq N$. We say that $\{a_n\}$ *converges in* $\mathbf{R}_{\text{ext}}$ if there exists $L \in \mathbf{R}_{\text{ext}}$ to which $\{a_n\}$ converges.

In the definition above, for $L \in \mathbf{R}$, "$\{a_n\}$ converges in $\mathbf{R}_{\text{ext}}$ to L" means "$\{a_n\}$ converges in $\mathbf{R}$ to L."

Exercise

5. Let $\{a_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathbf{R}_{\text{ext}}$. Show that there is at most one $L \in \mathbf{R}_{\text{ext}}$ such that $\{a_n\}$ converges in $\mathbf{R}_{\text{ext}}$ to L.

In view of the result of Exercise 5, we can define *the* limit of a sequence that converges in $\mathbf{R}_{\text{ext}}$:

Definition 2.2 Let $\{a_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathbf{R}_{\text{ext}}$ that converges in $\mathbf{R}_{\text{ext}}$. We define the *limit* of $\{a_n\}$ in $\mathbf{R}_{\text{ext}}$, written $\lim_{n\to\infty} a_n$, to be the unique $L \in \mathbf{R}_{\text{ext}}$ such that $\{a_n\}$ converges in $\mathbf{R}_{\text{ext}}$ to L.

Thus, for a sequence $\{a_n\}$ in $\mathbf{R}_{\text{ext}}$, and $L \in \mathbf{R}_{\text{ext}}$,

$$\lim_{n\to\infty} a_n = L \text{ if and only if } \{a_n\} \text{ converges in } \mathbf{R}_{\text{ext}} \text{ to } L. \tag{2.2}$$

We use the phrases "$\lim_{n\to\infty} a_n$ exists in $\mathbf{R}_{\text{ext}}$" and "$\{a_n\}$ converges in $\mathbf{R}_{\text{ext}}$" synonymously.

As stated at the beginning of these notes, we have introduced $\mathbf{R}_{\text{ext}}$ as a convenience for dealing with sequences in $\mathbf{R}$ (not $\mathbf{R}_{\text{ext}}$) that are potentially unbounded, and subsets of $\mathbf{R}$ that are potentially unbounded. We have allowed Definition 2.2 to cover more general sequences—sequences in $\mathbf{R}_{\text{ext}}$—not because we ultimately care about sequences in $\mathbf{R}_{\text{ext}}$, but because sequences in $\mathbf{R}_{\text{ext}}$ will arise when we define lim sup and lim inf of real-valued sequences later.

For a sequence $\{a_n\}$ in $\mathbf{R}$, we have previously (outside of these notes) defined what the notation "$\lim_{n\to\infty} a_n = L$" means for $L \in \mathbf{R}$, and, separately, what it means when L is the symbol ∞ or $-\infty$. The difference between equation (2.2) and our previous definition is one of *viewpoint*. Previously, we've said "We *write* '$\lim_{n\to\infty} a_n = \infty$' if for all $c \in \mathbf{R}$, there exists $N \in \mathbf{N}$ such that $a_n > c$ for all $n \geq N$." Such wording defines what the *inseparable string of notation* "$\lim_{n\to\infty} a_n = \infty$" means; it does *not* assign a meaning to "$\lim_{n\to\infty} a_n$". The equals-sign in "$\lim_{n\to\infty} a_n = \infty$" previously did *not* have the conventional meaning of "equals", i.e. that the expressions to the left and right of "=" are the same element of some *set*. In the limit-equals-infinity case, previously we did not regard $\lim_{n\to\infty} a_n$ as an element of *any* set; it was just part of the notation "$\lim_{n\to\infty} a_n = \infty$". In Definition 2.2 and in equation (2.2), however, we *do* regard $\lim_{n\to\infty} a_n$ as an element of some set ($\mathbf{R}_{\text{ext}}$) in the infinite-limit case.

This difference in viewpoint is reflected also in Definition 2.1. Previously we've said that "$\lim_{n\to\infty} a_n = \infty$" means that $\{a_n\}$ *does not converge* (or that the limit *fails to exist*), but that convergence fails in a very particular way. In equation 2.2, if $\lim_{n\to\infty} a_n = \infty$, the sequence still does not converge <u>in $\mathbf{R}$</u>. But $\mathbf{R}_{\text{ext}}$ contains an element we are calling ∞, and we have made a special definition (Definition 2.1) of what it means for a sequence to *converge* to this element. Such a definition is necessary because "converges" is a term we have previously defined only for a sequence *in a metric space*, and $\mathbf{R}_{\text{ext}}$ is not a metric space.

Nothing in these notes supersedes the terminology you learned in Calculus 2 for the cases "$\lim_{n\to\infty} a_n = \infty$" and "$\lim_{n\to\infty} a_n = -\infty$". For a sequence in $\mathbf{R}$ for which $\lim_{n\to\infty} a_n = \infty$, it is *still correct* to say that the sequence diverges, and it is *still correct* to say that the sequence diverges to ∞. "Diverges to ∞" means the same thing as "converges in $\mathbf{R}_{\text{ext}}$ to ∞."

Exercises

6. Let $\{a_n\}$ be a sequence in $\mathbf{R}$ that converges in $\mathbf{R}_{\text{ext}}$ to $L \neq 0$. Prove that there exists $N \in \mathbf{N}$ such that for all $n \geq N$, $a_n \neq 0$ and $\text{sgn}(a_n) = \text{sgn}(L)$.[2] In class we proved this for the case $L \in \mathbf{R}$, so you need only supply the proof for the cases $L = \infty$ and $L = -\infty$. (Note: the assertion in this problem would remain true if "sequence in $\mathbf{R}$" were replaced by "sequence in $\mathbf{R}_{\text{ext}}$"; I simply am not asking you to spend time on an additional generalization that is not of much use.)

7. Let $\{a_n\}$, $\{b_n\}$ be convergent sequences in $\mathbf{R}_{\text{ext}}$ (i.e. sequences in $\mathbf{R}_{\text{ext}}$ that converge in $\mathbf{R}_{\text{ext}}$), and assume that $a_n \leq b_n$ for all $n \in \mathbf{N}$. Prove that

$$\lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n \ .$$

8. Prove that every monotone sequence in $\mathbf{R}_{\text{ext}}$ converges in $\mathbf{R}_{\text{ext}}$. ("Monotone", as usual, means "monotone increasing or monotone decreasing", as defined using the extended relations "$\leq$", "$\geq$".) It is acceptable to prove this only for the increasing case, and to state that the decreasing case is similar.

9. Let $\{a_n\}_{n\in\mathbf{N}}$ be a sequence in $\mathbf{R}$. For $n \in \mathbf{N}$ define

$$\begin{aligned} a'_n &= \sup\{a_k : k \geq n\} \in \mathbf{R}_{\text{ext}}, \\ a''_n &= \inf\{a_k : k \geq n\} \in \mathbf{R}_{\text{ext}}. \end{aligned}$$

Show that $\{a'_n\}$ and $\{a''_n\}$ are monotone sequences in $\mathbf{R}_{\text{ext}}$. (Note: by hypothesis we are excluding ∞ and $-\infty$ as possible values of a_n, but we cannot exclude them as

[2] Generally the terminology we use for a statement of the form "there exists $N \in \mathbf{N}$ such that for all $n \geq N$, the statement $P(n)$ is true" is the simpler, less formal "$P(n)$ is true for n sufficiently large."

possible values of a'_n or a''_n. This is the sole reason that we allowed the sequence in Definition 2.2 to be $\mathbf{R}_{\rm ext}$-valued, rather than requiring it to be $\mathbf{R}$-valued; we needed a definition that would apply to the sequences $\{a'_n\}$ and $\{a''_n\}$ in all cases.)

10. (Notation as in Exercise 9). From Exercises 8 and 9, the sequences $\{a'_n\}$ and $\{a''_n\}$ always converge in $\mathbf{R}_{\rm ext}$. We define

$$\begin{aligned} \limsup_{n\to\infty} a_n &= \lim_{n\to\infty} a'_n, \\ \liminf_{n\to\infty} a_n &= \lim_{n\to\infty} a''_n. \end{aligned}$$

Prove that

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n,$$

with equality if and only if $\{a_n\}$ converges in $\mathbf{R}_{\rm ext}$.

3 Arithmetic in $\mathbf{R}_{\rm ext}$

Definition 3.1 For $c \in \mathbf{R}_{\rm ext} \setminus \{0\}$, we define the *sign of* c, written $\operatorname{sgn}(c)$, by

$$\operatorname{sgn}(c) = 1 \text{ if } c > 0, \quad \operatorname{sgn}(c) = -1 \text{ if } c < 0.$$

As mentioned earlier, we cannot usefully extend addition and multiplication to maps $\mathbf{R}_{\rm ext} \times \mathbf{R}_{\rm ext} \to \mathbf{R}_{\rm ext}$; we must restrict the domains of these binary operations. Similar restrictions also enter the definition of subtraction and division on $\mathbf{R}_{\rm ext}$. Since these operations *extend* operations on $\mathbf{R}$—i.e. we are not changing the definitions of $x+y, x-y, xy$, or x/y for $x, y \in \mathbf{R}$—the definition below is given only for cases in which at least one of the operands is $\pm\infty$. Also, instead of saying "the pair (x, y) is not in the domain of the map $+ : \mathbf{R}_{\rm ext} \times \mathbf{R}_{\rm ext} \to \mathbf{R}_{\rm ext}$", we simply say "We do not define $x+y$,", or "$x+y$ is not defined," or "$x+y$ is undefined." A similar comment applies to the other arithmetic operations.

Definition 3.2 1. *Addition*

(a) We define $x + \infty = \infty = \infty + x$ for all $x \in \mathbf{R} \bigcup \{\infty\}$.

(b) We define $x + (-\infty) = -\infty = (-\infty) + x$ for all $x \in \mathbf{R} \bigcup \{-\infty\}$.

We do not define "$\infty + (-\infty)$" *or* "$(-\infty) + \infty$". *In particular,* ∞ *and* $-\infty$ *do not have additive inverses.*

2. *Multiplication*

   (a) We define $1 \cdot x = x = x \cdot 1$ for all $x \in \mathbf{R}_{\text{ext}}$.

   (b) We define

$$(-1) \cdot \infty = -\infty = \infty \cdot (-1)$$

   and

$$(-1) \cdot (-\infty) = \infty = (-\infty) \cdot (-1).$$

   For convenience, we also define the notation "$-(-\infty)$" to mean ∞.[3] With this definition, we have $(-1) \cdot x = -x$ for all $x \in \mathbf{R}_{\text{ext}}$.

   (c) For all *nonzero* $c \in \mathbf{R}_{\text{ext}}$, we define

$$c \cdot \infty = \operatorname{sgn}(c) \cdot \infty = \infty \cdot c$$

   and

$$c \cdot (-\infty) = \operatorname{sgn}(c) \cdot (-\infty) = (-\infty) \cdot c$$

   (making use of the definition in (b) of multiplication by 1 and -1). We *do not define* "$\infty \cdot 0$", "$0 \cdot \infty$", "$(-\infty) \cdot 0$", or "$0 \cdot (-\infty)$".[4]

   Note that ∞ and $-\infty$ *do not have multiplicative inverses.*

   Just as for multiplication in $\mathbf{R}$, we often omit the symbol "$\cdot$" for multiplication (writing xy for $x \cdot y$) when no ambiguity can result.

3. *Subtraction*

   For $x, y \in \mathbf{R}_{\text{ext}}$, we define

$$x - y = x + (-y) \quad \textit{if the right-hand side is defined};$$

   otherwise we do not define $x - y$.

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[3] This does *not* follow from anything we've defined above, or earlier this semester. Recall that in a field $\mathbf{F}$, "$-x$" was *defined* to mean "the additive inverse of x" for $x \in \mathbf{F}$. But $\mathbf{R}_{\text{ext}}$ is not a field, and $-\infty$ does not have an additive inverse. Observe also that defining $-(-\infty) = \infty$ is defining *notation only*; it is not the definition of an *operation.*

[4] For students who've heard of measure theory, or may eventually take a course on measure theory: in measure theory it is common to define $0 \cdot (\pm\infty) = 0 = (\pm\infty) \cdot 0$. The rules for arithmetic on the extended reals are not unique; they are always defined within some context, and are chosen to simplify, or unify, the writing of certain statements *in that context.* The context for these notes is "sequences in $\mathbf{R}$", and arithmetic on $\mathbf{R}_{\text{ext}}$ has been defined so as to make Proposition 4.1 (later in these notes) true. Part 3 of this proposition would be false if we defined $0 \cdot \infty = 0$. In the context of measure theory, the convenience of having a definition of extended-real arithmetic that unifies certain measure-theoretic statements about quantities that could be either finite or infinite, outweighs the convenience of having a definition that unifies statements about sequences whose limits could be either finite or infinite.

4. *Division*

For all $c \in \mathbf{R}$, we define

$$\frac{c}{\infty} = 0 = \frac{c}{-\infty}\ .$$

We *do not define* $\frac{\infty}{\infty}$, $\frac{-\infty}{\infty}$, $\frac{\infty}{-\infty}$, or $\frac{-\infty}{-\infty}$. We *do not define* $\frac{x}{0}$ for *any* $x \in \mathbf{R}_{\text{ext}}$. (For example, "$\frac{1}{0}$" is undefined, even in $\mathbf{R}_{\text{ext}}$; it is *not* assigned the value ∞.) We also use the notation "x/y" for $\frac{x}{y}$", just as in $\mathbf{R}$.

Observe that for $x, y \in \mathbf{R}_{\text{ext}}$, $x+y$ is defined if and only if $y+x$ is defined, and the two quantities are equal when defined. A similar comment applies to multiplication. Thus these operations are "as commutative as they can be."

However, note that this commutativity is a consequence of our definitions; it does *not* follow from the fact that the symbols "+" and "·" are used for these operations. Similarly, you cannot assume that other properties of real arithmetic that you're familiar with—e.g. associativity of addition and multiplication, and the distributive law—apply in $\mathbf{R}_{\text{ext}}$. Any such property must be proven true before you can use it, and not all these properties are true in $\mathbf{R}_{\text{ext}}$. Fortunately, we rarely come across an instance in which this matters. Nonetheless, do the following two exercises:

Exercises

11. Let $x, y, z \in \mathbf{R}_{\text{ext}}$. Consider the (not necessarily valid) equation

$$(x + y) + z = x + (y + z). \tag{3.3}$$

(Here it is understood that if $x + y$ is undefined, then so is $(x + y) + z$; a similar comment applies to the right-hand side.) Show the following: (i) If the set $\{x, y, z\}$ contains *both* ∞ and $-\infty$, then *neither* side of (3.3) is defined. (ii) If the set $\{x, y, z\}$ does *not* contain both ∞ and $-\infty$, then both sides of (3.3) are defined, and the two sides are equal.

Thus, addition in $\mathbf{R}_{\text{ext}}$ is "reasonably associative" in $\mathbf{R}_{\text{ext}}$: if one side of (3.3) is defined, then so is the other, and the two sides are equal.

12. (Multiplicative analog of the previous exercise.) Let $x, y, z \in \mathbf{R}_{\text{ext}}$. Consider the (not necessarily valid) equation

$$(xy)z = x(yz). \tag{3.4}$$

(Here it is understood that if xy is undefined, then so is $(xy)z$; a similar comment applies to the right-hand side.) Show the following: (i) If the set $\{x, y, z\}$ contains *both* 0 and either ∞ or $-\infty$, then *neither* side of (3.4) is defined. (ii) In all other cases, both sides of (3.3) are defined, and the two sides are equal.

Thus, multiplication in $\mathbf{R}_{\text{ext}}$ is "reasonably associative" in $\mathbf{R}_{\text{ext}}$, in the same sense as for addition.

In contrast, the (would-be) distributive law behaves very badly in $\mathbf{R}_{\mathrm{ext}}$. For example, consider the (not necessarily valid) equation

$$\infty \cdot (1-2) \stackrel{?}{=} \infty \cdot 1 - \infty \cdot 2. \tag{3.5}$$

The left-hand side is perfectly well-defined; it equals $\infty \cdot (-1)$, hence $-\infty$. On the right-hand side, both $\infty \cdot 1$ and $\infty \cdot 2$ are defined; they both equal ∞. But $\infty - \infty$ is not defined, so the right-hand side of (3.5) is not defined. So, in $\mathbf{R}_{\mathrm{ext}}$, it is possible for one side of the (not necessarily valid) equation "$c(x+y) = cx + cy$" to be defined without the other side being defined.

It is not worth the trouble to classify the triples $(c, x, y) \in \mathbf{R}_{\mathrm{ext}} \times \mathbf{R}_{\mathrm{ext}} \times \mathbf{R}_{\mathrm{ext}}$ *for which either both sides of "$c(x+y) = cx + cy$" are defined or neither side is defined, nor is it worth the trouble to figure out whether both side are equal when both are defined.* Exercise 11 and 12 above are not particularly important in their own right. Rather, the reason that they've been included in these notes, alongside the example above concerning distributivity, is to help make you conscious of the following: (i) *there is no such thing as "proof by notation"*; and (ii) just because some generalization of some statement is true, you cannot "reason by analogy" that some other similar-in-spirit generalization is true. Unlike establishing two analogous facts by analogous proofs, "reasoning by analogy" is not a method of proof; it's an excuse for *not* taking the trouble to try to prove something.

4 $\mathbf{R}_{\mathrm{ext}}$ and the arithmetic of sequences in $\mathbf{R}$

As mentioned earlier, the reason for introducing $\mathbf{R}_{\mathrm{ext}}$ is so that certain infinite-limit statements that are *analogous* to finite-limit statements can be written efficiently, combining the finite-limit and infinite-limit cases into a single statement. The definition of arithmetic in $\mathbf{R}_{\mathrm{ext}}$ was chosen to make the proposition below true. In this proposition, if we allowed the sequences $\{a_n\}, \{b_n\}$ to lie in $\mathbf{R}_{\mathrm{ext}}$, rather than requiring them to lie in $\mathbf{R}$, all the conclusions would still be true. The reason we have not stated the proposition in this greater generality is that our purpose here is *not* to state or prove anything about sequences in $\mathbf{R}_{\mathrm{ext}}$, except as a tool to state or prove something about real-valued sequences. The extended reals are *only* a means to this end.

Proposition 4.1 *Let* $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ *be sequences in* $\mathbf{R}$ *that converge in* $\mathbf{R}_{\mathrm{ext}}$*. Then the following are true. (All limits below are considered to be limits in* $\mathbf{R}_{\mathrm{ext}}$*.)*

1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ <u>if</u> the right-hand side of this equation is defined.

2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$ <u>if</u> the right-hand side of this equation is defined.

3. $\lim_{n\to\infty}(a_n b_n) = (\lim_{n\to\infty} a_n)\cdot(\lim_{n\to\infty} b_n)$ <u>if</u> the right-hand side of this equation is defined.

4. $\lim_{n\to\infty}(a_n/b_n) = (\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$ <u>if</u> the right-hand side of this equation is defined.

Note that in statement 4, we do not have to insert the assumption "$\lim_{n\to\infty} b_n \neq 0$" explicitly, since if "$\lim_{n\to\infty} b_n = 0$" then $(\lim_{n\to\infty} a_n)/(\lim_{n\to\infty} b_n)$ is not defined. However, in statement 4, we *are* using the result of Exercise 6, and the convention that if $\{b_n\}$ is a sequence for which b_n is nonzero for all n sufficiently large, but is zero for some values of n, then the sequence $\{a_n/b_n\}$ is considered to have initial index n_0, where n_0 is any index for which $b_n \neq 0$ for all $n \geq n_0$.

Exercises

13. Prove Proposition 4.1.

14. One case of part 1 of Proposition 4.1 is the case in which $\{a_n\}$ converges in $\mathbf{R}$ and $\{b_n\}$ converges in $\mathbf{R}_{\text{ext}}$ to ∞ or $-\infty$. Prove the following generalization of this case: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are sequences in $\mathbf{R}$, with $\{a_n\}$ bounded (but not necessarily convergent) and $\{b_n\}$ converging in $\mathbf{R}_{\text{ext}}$ to ∞ or $-\infty$, then $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} b_n$ $(= \pm\infty)$.

15. One case of part 3 of Proposition 4.1 is the case in which $\{a_n\}$ converges in $\mathbf{R}$. Prove the following generalization of this case: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are sequences in $\mathbf{R}$, with $\liminf\limits_{n\to\infty}|a_n| > 0$ and with all a_n having the same sign $s \in \{\pm 1\}$ for n sufficiently large, and with $\{b_n\}$ converges in $\mathbf{R}_{\text{ext}}$ to ∞ or $-\infty$, then $\lim_{n\to\infty}(a_n b_n) = s\cdot\lim_{n\to\infty} b_n$ $(= \pm\infty)$.