MAA 4211, Fall 2017—Assignment 3's non-book problems

B1. (a) Let Z be any nonempty set, and let $\text{Fun}(Z, \mathbf{R})$ denote the set of all functions $Z \to \mathbf{R}$. Temporary notation, just for this problem: let $\mathbf{0}$ denote the constant function with value 0 (i.e. $\mathbf{0}(z) = 0$ for all $z \in Z$). For $f, g \in \text{Fun}(Z, \mathbf{R})$ and $c \in \mathbf{R}$ we define $f + g \in \text{Fun}(Z, \mathbf{R})$ and $cf \in \text{Fun}(Z, \mathbf{R})$ by

$$\begin{aligned} f+g &= \text{the function } z \mapsto f(z) + g(z), \\ cf &= \text{the function } z \mapsto cf(z). \end{aligned}$$

Check that, with the operations above, $\text{Fun}(Z, \mathbf{R})$ is a vector space with zero-element $\mathbf{0}$.

(b) Let $\mathbf{R}^\infty$ denote the set of all functions $\mathbf{N} \to \mathbf{R}$. For $f \in \mathbf{R}^\infty$, one of the notations we commonly use is $(x_1, x_2, x_3, \ldots)$, where $x_n = f(n), n \in \mathbf{N}$. Thus an element of $\mathbf{R}^\infty$ is also called an *infinite sequence* in $\mathbf{R}$. By part (a), $\mathbf{R}^\infty$ is a vector space. Check that, in the sequence-notation above, the operations and zero-element in $\mathbf{R}^\infty$ (as defined in part (a)) are given by

$$\begin{aligned} (x_1, x_2, x_3, \ldots) + (y_1, y_2, y_3, \ldots) &= (x_1 + y_1, x_2 + y_2, x_3 + y_3, \ldots), \\ c(x_1, x_2, x_3, \ldots) &= (cx_1, cx_2, cx_3, \ldots), \\ \mathbf{0} &= (0, 0, 0, \ldots). \end{aligned}$$

(c) A real-valued sequence $\vec{x} = (x_1, x_2, x_3, \ldots) \in \mathbf{R}^\infty$ (switching from the notation "f" in part (b)) is called *bounded* if there exists $M \in \mathbf{R}$ such that $|x_n| \leq M$ for all $n \in \mathbf{N}$; equivalently, if the set $\{|x_n| : n \in \mathbf{N}\}$ is bounded above. Let $\mathbf{R}_b^\infty \subset \mathbf{R}^\infty$ denote the set of bounded real-valued sequences. Show that $\mathbf{R}_b^\infty$ is a vector subspace of $\mathbf{R}^\infty$.

(d) For any $\vec{x} \in \mathbf{R}_b^\infty$, the set $\{|x_n| : n \in \mathbf{N}\}$ is nonempty and bounded above, hence has a least upper bound. Therefore we can define a function $\| \ \|_\infty : \mathbf{R}_b^\infty \to \mathbf{R}$ by

$$\|\vec{x}\|_\infty := \sup\{|x_n| : n \in \mathbf{N}\}.$$

Show that $\| \ \|_\infty$ is a norm on the vector space $\mathbf{R}_b^\infty$. (Note: the "∞" subscript in $\| \ \|_\infty$ has nothing to do with the "∞" superscript in $\mathbf{R}^\infty$. Rather, the notation for this norm comes from the analogous norm on $\mathbf{R}^n$, where we can replace "sup" by "max".) We call this norm the *ℓ^∞-norm* or *sup-norm* on $\mathbf{R}_b^\infty$.

(e) Let d_∞ denote the metric on $\mathbf{R}_b^\infty$ associated with the ℓ^∞ norm; we call d_∞ the *ℓ^∞ metric* on $\mathbf{R}_b^\infty$. Check that d_∞ is given by

$$d_\infty(\vec{x}, \vec{y}) = \sup\{|x_n - y_n| : n \in \mathbf{N}\}.$$

B2. Let $(V, \| \ \|)$ be a normed vector space, viewed as a metric space with the associated metric. Show that for all $v \in V$,

$$\begin{aligned}\text{for each } r > 0 \text{ we have} \quad B_r(v) &= \{v + w \mid w \in B_r(0)\},\\ \text{and for each } r \geq 0 \text{ we have} \quad \overline{B}_r(v) &= \{v + w \mid w \in \overline{B}_r(0)\}.\end{aligned}$$

In other words, each open (respectively, closed) ball centered at a given v is simply the translation, by v, of the open (respectively, closed) ball of the same radius centered at the origin.

B3. Define a metric d on the set of rational numbers $\mathbf{Q}$ by $d(x, y) = |x-y|$ (the restriction to $\mathbf{Q}$ of the standard metric on $\mathbf{R}$). Give an example, with proof, of a nonempty, proper subset of $(\mathbf{Q}, d)$ that is both open and closed in this metric space. (Do not expect your subset to be either open or closed in $\mathbf{R}$, let alone *both* open and closed in $\mathbf{R}$. There is no nonempty, proper subset of $\mathbf{R}$ that is both open and closed with respect to the standard metric.)

B4. Let $n \geq 1$ and let $\mathbf{E}^n$ denote Euclidean n-space. Let $p \in \mathbf{E}^n$, $r \geq 0$. Let $\overline{B}_r(p)$ denote the closed ball of radius r centered at p. Prove that $\overline{B}_r(p)$ is not an open set.

Remember: (i) "Closed" does not imply "not open". The fact that a closed ball in a metric space is a closed set does not imply that a closed ball can't be an open set. (In fact, in one of the Rosenlicht problems you will see an example in which *every* ball is simultaneously an open set and a closed set.) (ii) There is no such thing as "proof by picture". If you are asserting, for example, that a certain open ball contains points of some other set, you have to *prove* that assertion, not merely assert that it's true based on some picture you've drawn and your intuition.

B5. Let (E, d) be a metric space. For purposes of this problem, for each $p \in E$ define a property we'll call "boundedness with respect to p" as follows: a set $S \subset E$ is *bounded with respect to* p if S is contained in some ball *centered at* p.

Let $p \in E$. Show that for every $S \subset E$, the following are equivalent:

(i) S is bounded with respect to p.

(ii) S is bounded.

(iii) S is bounded with respect to q for all $q \in E$.

B6. Let $(E, d) = \mathbf{E}^2$ (Euclidean 2-space). Let $p \in E$ and let $r > 0$.

(a) Show that $\overline{B_r(p)} = \overline{B}_r(p)$ (i.e. the closure of an open ball is the closed ball with the same center and radius).

(b) Show that $\partial B_r(p)$ is the *sphere* of radius r centered at p, defined as $\{q \in E \mid d(p, q) = r\}$. (This is the general definition of "sphere" for an arbitrary metric space; spheres in $\mathbf{E}^2$ are circles.)

(c) Re-do parts (a) and (b) with $\mathbf{E}^2$ replaced by $\mathbf{E}^n$, where n is arbitrary. Once (a) and (b) are done, you should find this easy; if not, then your arguments in (a) and (b) are probably wrong.

B7. Give an example of a metric space E in which there is an open ball $B_r(p)$ whose closure is *not* the closed ball $\overline{B}_r(p)$. (You have already encountered a metric space with this property.)

B8. (a) Let (E, d) be a metric space, $p \in E$, $r > 0$. Let $S_r(p)$ denote the sphere of radius r centered at p (see B6(b)). Prove that $\partial(B_r(p)) \subset S_r(p)$.

(b) Give an example of a metric space (E, d) in which there is an open ball $B_r(p)$ for which $\partial(B_r(p)) \neq S_r(p)$.