MAA 4211, Fall 2017—Assignment 4's non-book problems

B1. Let (E, d) be a metric space, let $(p_n)_{n=1}^{\infty}$ be a sequence in E, and define sequences $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ by

$$\begin{aligned} x_n &= p_{2n-1} \text{ for each } n \in \mathbf{N}, \\ y_n &= p_{2n} \text{ for each } n \in \mathbf{N}. \end{aligned}$$

(In other words, (x_n) and (y_n) are the subsequences of (p_n) given by the odd-numbered terms and even-numbered terms, respectively.) Prove that the following are equivalent:

(i) $(p_n)_{n=1}^{\infty}$ converges.

(ii) Both $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ converge, and their limits are equal.

Prove also that if condition (ii) holds, then $\lim_{n\to\infty} p_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

B2. Let (E, d) be a metric space, let $(p_n)_{n=1}^{\infty}$ be a Cauchy sequence in E, and assume that this sequence has a convergent subsequence $(p_{n_i})_{i=1}^{\infty}$. Let $p = \lim_{i\to\infty} p_{n_i}$. Show that the original sequence $(p_n)_{n=1}^{\infty}$ also converges to p.

(Note (E, d) is not assumed to have any properties other than being a metric space; e.g. we are not assuming (E, d) is complete. The hypotheses say only that *this particular* Cauchy sequence $(p_n)_{n=1}^{\infty}$ has a convergent subsequence, not that every sequence has a convergent subsequence, and not that every Cauchy sequence has a convergent subsequence.)

B3. Let d_1 and d_2 be two metrics on a (nonempty) set E. Call a set $S \subset E$ "d_1-open" if it is open in the metric space (E, d_1), and "d_2-open" if it is open in the metric space (E, d_2). Analogously define "d_i-bounded set", "d_i-Cauchy sequence", and "d_i-convergent sequence".

(a) Suppose that there exists $c > 0$ such that $d_2(p, q) \le c d_1(p, q)$ for all $p, q \in E$. Prove that every d_2-open subset of E is d_1-open. (*Note:* This was already proven in class, but you should re-prove it anyway to make sure you understand the argument.)

(b) Recall from class that metrics d_1, d_2 on a set E are called *equivalent* if there exist $c_1, c_2 > 0$ such that for all $p, q \in E$, $d_2(p, q) \le c_1 d_1(p, q)$ and $d_1(p, q) \le c_2 d_2(p, q)$. Below, we write "$d_1 \sim d_2$" for "$d_1, d_2$ are equivalent metrics" (on a given set).

Let E be an arbitrary nonempty set. Prove the following:

(i) The relation $\sim$ is an equivalence relation on the set of all metrics on E.

(ii) Equivalent metrics on E determine the same open sets and the same closed sets. I.e. if d_1 and d_2 are equivalent and $U \subset E$, then U is d_1-open iff U is d_2-open, and U is d_1-closed iff U is d_2-closed.

(iii) Equivalent metrics determine the same bounded sets. I.e. if d_1 and d_2 are equivalent and $U \subset E$, then U is d_1-bounded if U is d_2-bounded.

(iv) Equivalent metrics determine the same Cauchy sequences, the same convergent sequences, and the same limits of convergent sequences. I.e. if d_1 and d_2 are equivalent, $q \in E$, and $(p_n)_{n=1}^{\infty}$ is a sequence in E, then (p_n) is d_1-Cauchy iff (p_n) is d_2-Cauchy in (E, d_2), and (p_n) is d_1-convergent to q iff (p_n) is d_2-convergent to q.

B4. Let $\| \ \|$ and $\| \ \|'$ be two norms on a vector space V. We call these two norms *equivalent* if there exist $c_1, c_2 > 0$ such that for all $v \in V$, $\|v\| \le c_1\|v\|'$ and $\|v\|' \le c_2\|v\|$.[1] Prove that if norms $\| \ \|$ and $\| \ \|'$ are equivalent, then their associated metrics are equivalent.

B5. Let $n \in \mathbf{N}$. Prove that the ℓ^1, ℓ^2, and ℓ^∞ norms on $\mathbf{R}^n$ are all equivalent to each other (i.e. each is equivalent to the other two), and hence that their associated metrics are equivalent to each other.

B6. Let (E_1, d_1) and (E_2, d_2) be metric spaces. In earlier homework (Rosenlicht, p. 61/1c) you showed that the function $d : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

is a metric on $E_1 \times E_2$. (a) Show that the function $d' : (E_1 \times E_2) \times (E_1 \times E_2) \to \mathbf{R}$ defined by

$$d'((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2),$$

is also a metric on $E_1 \times E_2$.

(b) Show that the metrics d and d' on $E_1 \times E_2$ are equivalent. (Note: "Show" always means "Prove".)

(c) Show that if (E_1, d_1) and (E_2, d_2) are complete, then so are $(E_1 \times E_2, d)$ and $(E_1 \times E_2, d')$.

B7. In previous homework you showed that $\mathbf{R}_b^\infty$, the space of bounded real-valued sequences, is a vector subspace of $\mathbf{R}^\infty$, the space of all real-valued sequences. Now define

$$\mathbf{R}_0^\infty = \{\vec{a} = (a_m)_{m=1}^{\infty} \in \mathbf{R}^\infty : a_m = 0 \text{ for all but finitely many } m\}.$$

[1]Some mathematicians use the term "strongly equivalent" for *metrics* related to each other as in problem 5. What these mathematicians call "equivalent metrics" is what I call "*topologically* equivalent metrics". However, for *norms* related to each other as in problem 6, there is universal agreement on the terminology "equivalent".

(a) Show that $\mathbf{R}_0^\infty$ is a vector subspace of $\mathbf{R}_b^\infty$.

(b) In view of part (a), we can restrict the ℓ^∞ norm on $\mathbf{R}_b^\infty$, as well as its associated metric d_∞, to $\mathbf{R}_0^\infty$. (Note that for $\vec{a} \in \mathbf{R}_0^\infty$, we can actually write $\|\vec{a}\|_\infty = \max\{|a_m| : m \in \mathbf{N}\}$ instead of $\|\vec{a}\|_\infty = \sup\{|a_m| : m \in \mathbf{N}\}$; "max" is well-defined here, since for each $\vec{a} \in \mathbf{R}_0^\infty$, the set $\{|a_m| : m \in \mathbf{N}\}$ is finite.) Show that $(\mathbf{R}_0^\infty, d_\infty)$ is not complete. I.e. produce a sequence $(\vec{a}^{(i)})_{i=1}^\infty$ in $\mathbf{R}_0^\infty$ that is Cauchy but does not converge in $(\mathbf{R}_0^\infty, d_\infty)$. (Hint: consider sequences $(\vec{a}^{(i)})_{i=1}^\infty$ in which $a_j^{(i)} = 0$ for $j > i$ but $a_i^{(i)} \neq 0$, and for which the only difference between $\vec{a}^{(i)}$ and $\vec{a}^{(i-1)}$ is the i^{th} term.)