MAA 4211, Fall 2017—Assignment 6's non-book problems

Problems B1 and B2(c) are intended to help you better relate the formal meaning of "connected subset of a metric space" to the intuitive notion of what it sounds like this terminology ought to mean.

B1. Let (E, d) be a metric space, $S \subset E$. Prove that the following are equivalent:

(i) S is not connected.

(ii) There exist nonempty subsets $A, B \subset S$ such that $S = A \bigcup B$ and $\overline{A} \bigcap B = \emptyset = A \bigcap \overline{B}$. (Here $\overline{A}$ and $\overline{B}$ denote the closures of A and B in E, not in the subspace (S, d).)

Some things to note: (1) Under the conditions on sets A, B in (ii), we automatically have $A \bigcap B = \emptyset$, so $S = A \coprod B$. For arbitrary subsets $A, B \subset E$, the condition "$\overline{A} \bigcap B = \emptyset$" is *stronger* (more restrictive) than "$A \bigcap B = \emptyset$." (2) In (ii), we are *not* assuming that A and B are open in (S, d). (Openness in (S, d) will end up being a *consequence* of what we've assumed, but it's not one of our assumptions.) (3) The equivalence of (i) and (ii) is interesting only for *proper* subsets $S \subsetneq E$. When $S = E$, the equivalence follows immediately from the definition of "connected metric space".

B2. Let (E, d) be a metric space, $S \subset E$ a nonempty subset, and $p \in E$. The *distance from p to S*, which we will write as $\text{dist}(p, S)$, is defined to be $\inf\{d(p, q) \mid q \in S\}$.

(a) Prove that $\text{dist}(p, S) = 0$ if and only if $p \in \overline{S}$. You may use any facts stated in the Interiors, Closures, and Boundaries handout.

(b) For $(E, d) = \mathbf{E}^2$, give an example of each of the following.

(i) A subset S and a point $p \notin S$ for which the infimum defining $\text{dist}(p, S)$ *is not* achieved.

(ii) A subset S and a point $p \notin S$ for which the infimum defining $\text{dist}(p, S)$ *is* achieved. (Note that "The infimum defining $\text{dist}(p, S)$ is achieved" is equivalent to "There is a point $q \in S$ that, among all points in S, minimizes distance to p.")

(c) For purposes of this problem, call S *non-connected* if S is not connected.[1] Using part (a) and the result of B1, prove that S is non-connected if and only if $S = A \bigcup B$

[1]Although it is tempting to use the term "disconnected" for "not connected", topologists generally don't do this, instead reserving "disconnected" as one piece of the terminology for topological (sub)spaces that fail in some spectacular way to be connected, such as *totally disconnected* spaces (see problem B4(g)). The most common terminology for "not connected" is "not connected", not "non-connected". Here I am using the term "non-connected" because a sentence of the form "S is not connected if and only if ..." is ambiguous—does it mean that "not connected" is equivalent to the specified conditions, or does it mean "The statement 'S is connected if and only if ...' " is false? Make sure you avoid this sort of ambiguous phrasing in your answers.

for some nonempty sets $A, B \subset S$ for which every point of each set is a positive distance from the other set (i.e. $\text{dist}(p, B) > 0$ for all $p \in A$, and $\text{dist}(p, A) > 0$ for all $p \in B$).

Motivation for part (c): Recall that, heuristically, we wanted "S is not connected" to mean that S cannot be partitioned into two nonempty disjoint subsets that "don't touch each other". There is no official definition of one subset of a metric space *touching*, or not touching, another. However, were we (not unreasonably) to define "A does not touch B" to mean "every point of A is a positive distance from B", then the characterization of non-connectedness in this problem would turn the heuristic characterization of "not connected" into a precise one that agrees with the mathematical definition.

B3. Let (E, d) be a metric space, $S \subset E$ a connected subset. Prove that the closure of S is connected.

B4. (This problem gives a second proof of a lemma we proved in class.) Let (E, d) be a metric space, let $(p_n)_{n=1}^{\infty}$ be a convergent sequence in E, let $p = \lim_{n\to\infty} p_n$, let $N \in \mathbf{N}$, and let $\epsilon > 0$. Assume that for all $n, m \geq N$, $d(p_n, p_m) < \epsilon$. Use the appropriate half of the "sequential characterization of continuity" to prove that for all $n \geq N$, $d(p_n, p) \leq \epsilon$. *Hint*: Consider the function $f : E \to \mathbf{R}$ defined by $f(q) = d(p_n, q)$.

B5. Let X, Y be nonempty sets, let d_X, d'_X be equivalent metrics on X, and let d_Y, d'_Y be equivalent metrics on Y. Prove that if a function $f : X \to Y$ is continuous as a map $(X, d_X) \to (Y, d_Y)$, then f is continuous as a map $(X, d'_X) \to (Y, d'_Y)$. (In particular this holds if we vary only one of the metrics, i.e. if $d_X = d'_X$ or $d_Y = d'_Y$.)

B5. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $f : (X, d_X) \to (Y, d_Y)$ a continuous function, and $S \subset X$ nonempty. Prove that the restriction $f|_S$ is continuous as a function $(S, d_X) \to (Y, d_Y)$.

B6. In Rosenlicht p. 91/#3, suppose you remove the hypothesis that the sets S_1, S_2 are both closed. Is the conclusion still true? (Prove your answer, of course.)