(12.1) 2017

<u>Digression for Euclidean norm on $\mathbb{R}^n$</u> : Norm assoc'd with an inner product

<u>Def</u>: Let V be a (real) $\overbrace{\text{vector space}}^{\text{v.s.}}$. An <u>inner product</u> or <u>scalar product</u> on V is a map
$$V \times V \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto \langle v, w \rangle$$

that is <u>positive-definite</u>, <u>symmetric</u>, and <u>bilinear</u>

positive-definite means: $\langle v, v \rangle \geq 0 \ \forall v$, and $= 0$ iff $v = 0$

$0_{\mathbb{R}}$ $\qquad$ 0_V

symmetric means: $\langle v, w \rangle = \langle w, v \rangle$

bilinear means:

$\langle cv, w \rangle = c\langle v, w \rangle = \langle v, cw \rangle$ $\quad (\forall c \in \mathbb{R},\ v \in V,\ w \in V)$

and

$\langle v_1 + v_2, w \rangle = \langle v_1, w \rangle + \langle v_2, w \rangle$

$\langle v, w_1 + w_2 \rangle = \langle v, w_1 \rangle + \langle v, w_2 \rangle$.

<u>Ex.</u> $V = \mathbb{R}^n$, $\langle x, y \rangle = \sum_i x_i y_i$ (where $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$).

This is called the <u>standard</u> or <u>Euclidean</u> inner prod. on $\mathbb{R}^n$.
(HW: Check that this <u>is</u> an inner product.)

<u>Lemma</u>: Let $\langle\ ,\ \rangle$ be an inner prod. on v.s. V. Then
$$\langle 0_V, w \rangle = 0_{\mathbb{R}} \quad \forall w \in V.$$

<u>Pf</u>: Let $w \in V$. Then
$$\langle 0_V, w \rangle = \langle 0_{\mathbb{R}} 0_V, w \rangle = 0_{\mathbb{R}} \langle 0_V, w \rangle = 0_{\mathbb{R}} \quad . \ \text{QED}$$

Usually I won't put subscripts on different 0's. I'll only do it when you can't easily tell from context what "0" means.

<u>Note</u>: By symmetry of inner prod., the lemma implies $\langle w, 0_V \rangle = 0_{\mathbb{R}}$ as well

Given an inner prod. $\langle\, ,\, \rangle$ on v.s. V, define

$$\|v\| = \sqrt{\langle v, v\rangle}.$$

<u>Prop</u>: $\|\ \|$ is a norm on V.

<u>Pf</u>: (i) "$\|v\| \geq 0 \quad \forall v$" is clear. ($\checkmark$)

(ii) $\|v\| = 0 \Leftrightarrow \langle v, v\rangle = 0 \Leftrightarrow v = 0$. ($\checkmark$)

(iii) $\|cv\| = \sqrt{\langle cv, cv\rangle}$

$= \sqrt{c^2 \langle v, v\rangle}$

$= \sqrt{c^2}\sqrt{\langle v, v\rangle}$ (HW: Show $\sqrt{AB} = \sqrt{A}\sqrt{B}$ for $A, B \geq 0$)

$= |c|\,\|v\|$. (HW: show $\sqrt{c^2} = |c|$, $\forall c \in \mathbb{R}$.) $\checkmark$

(iv) Triangle ineq.

Postponed; need <u>Cauchy</u>-<u>Schwarz</u> <u>inequality</u>, discussed below.

Lemma (Cauchy-Schwarz inequality): For all $v, w \in V$, we have

$$|\langle v,w\rangle| \le \|v\|\,\|w\|, \qquad (*)$$

with equality iff. the set $\{v,w\}$ is linearly dependent.

Pf: Let $v, w \in V$.

Case 1: Assume $\{v,w\}$ lin. indep. Then $v \neq 0 \neq w$, and, $\forall t \in \mathbb{R}$, have $v + tw \neq 0$. From the latter,

$$f(t) \underset{\text{def.}}{=} \|v+tw\|^2 > 0, \quad \forall t \in \mathbb{R}.$$

Using bilinearity and symmetry, compute

$$\begin{aligned}\|v+tw\|^2 &\underset{\text{(by def.)}}{=} \langle v+tw, v+tw\rangle = \langle v,v\rangle + \langle v,tw\rangle + \langle tw,v\rangle + \langle tw,tw\rangle \\ &= \langle v,v\rangle + 2t\langle v,w\rangle + t^2\langle w,w\rangle \\ &= \|v\|^2 + 2t\langle v,w\rangle + t^2\|w\|^2\end{aligned}$$

Let $t_1 = -\dfrac{\langle v,w\rangle}{\|w\|^2}$. Then

$$\begin{aligned}0 < f(t_1) &= \|v\|^2 - 2\frac{\langle v,w\rangle^2}{\|w\|^2} + \frac{\langle v,w\rangle^2}{\|w\|^4}\|w\|^2 \\ &= \|v\|^2 - \frac{\langle v,w\rangle^2}{\|w\|^2},\end{aligned}$$

$$\Rightarrow \langle v,w\rangle^2 < \|v\|^2\|w\|^2,$$

$$\Rightarrow |\langle v,w\rangle| < \|v\|\,\|w\|.$$

<u>Case 2</u> : Assume $\{v, w\}$ lin. dep. If $w = 0 \; (= 0_V)$, then both sides of $(*)$ are $0 \; (= 0_{\mathbb{R}})$ (by Lemma on p. 12.1). If $w \neq 0$, then $v = cw$ for some $c \in \mathbb{R}$. We then have

$$\|v\| = \|cw\| = |c| \, \|w\| \qquad \text{(by (iii) on p. 12.2)}$$

and

$$|\langle v, w\rangle| = |\langle cw, w\rangle| = |c\langle w, w\rangle| = |c| \, |\langle w, w\rangle| = |c| \, \|w\|^2$$
$$= |c| \, \|w\| \, \|w\|$$
$$= \|v\| \, \|w\|.$$

Combining cases 1 and 2:

$$|\langle v, w\rangle| < \|v\| \, \|w\| \quad \text{if } \{v, w\} \text{ lin. indep.}$$

and

$$|\langle v, w\rangle| = \|v\| \, \|w\| \quad \text{if } \{v, w\} \text{ lin. dep.}$$

$\boxtimes$

~~Completing proof.~~

<u>Return to proof of Prop. on p. 12.2.</u> Only (iv), the Δ ineq., remains to be shown. Let $v, w \in V$. Then

$$\|v+w\|^2 = \langle v+w, v+w\rangle$$
$$= \langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle \qquad \text{(expanding as in pf. of C-S ineq.)}$$
$$= \|v\|^2 + 2\langle v, w\rangle + \|w\|^2$$
$$\leq \|v\|^2 + 2\|v\| \, \|w\| + \|w\|^2 \quad \text{by C-S ineq.}$$
$$= (\|v\| + \|w\|)^2.$$

Hence $\|v+w\| \leq \|v\| + \|w\|$. $\boxtimes$

We have now shown that the fcn $v \mapsto \|v\|$ def'd on p 12.2 is a norm. We call it the norm associated with (or defined by) the inner product.

<u>Cor</u>: The Euclidean norm $\| \|_2$ on $\mathbb{R}^n$ is, indeed, a norm.

<u>Pf</u>: The norm $\| \|_2$ is simply the norm associated with the standard inner prod. on $\mathbb{R}^n$. Hence ~~it follows that~~ it is, indeed, a norm. ☒

Some additional facts about metrics

(In general, not just on normed vector spaces.)

Let (E, d) be a metric space. Then

① $d(x, z) \le d(x, y_1) + d(y_1, y_2) + \cdots + d(y_{n-1}, y_n) + d(y_n, z)$

for all $x, z, y_1, \ldots, y_n \in E$.

Call this the <u>iterated</u> <u>triangle</u> <u>inequality</u>.

<u>Pf</u>: Δ ineq. + induction. (See Rosenlicht, p. 37). ☒

② For all $x, y, z \in E$,

$$|d(x, y) - d(x, z)| \le d(y, z)$$

<u>Pf</u>: See Rosenlicht, p. 37. ☒